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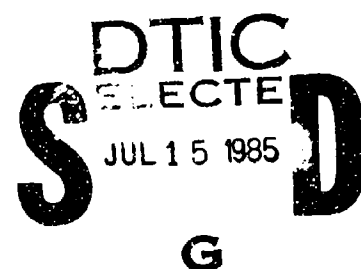
A General Theory of Target Identification: An Analytical Approach to Cognition, Perception, and Knowledge

J RICHARD HUYNEN

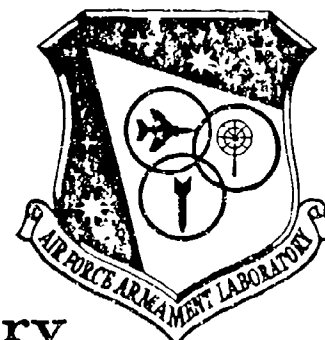
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PREFACE

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The Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service (NTIS), where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Franklin M. Gay

FRANKLIN M. GAY
Technical Director
Advanced Seeker Division

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SECTION I

OVERVIEW

At present many, techniques exist which aim at target identification. The traditional approach is to develop a data bank of target features and to store these in memory. For a new target, its features are measured and compared to those of the targets stored in memory. A best fit approach will select the target which most likely corresponds to the new target. Thus, the target has been identified.

This approach closely resembles the state of the art in language translation by machine in the 1960's. A dictionary look-up procedure provided a crude, word-to-word translation of the foreign language into English. These efforts were only partially successful. Later on, improvements were sought by adding connecting principles between words and between word phrases. Aside from these syntactical improvements, it was felt necessary to formalize meaning into sentence structure and interpretation. A distinction was made between syntactic surface structure and semantic deep structure. All of these efforts form the core of present machine translation techniques.

It seems profitable in the field of target identification to keep the linguistic model in mind, because there are many resemblances. First it is obvious that not all target features can be independent of each other; otherwise, no relationships can be formed between size, strength, shape, etc., of target structure. Also, targets may have a semantic aspect. A target may be identified by its meaning as well as by its observable features. An object propelled in my direction requires my attention even before I can recognize its features in detail. The attention focus is directed to the object because of a potential threat it poses. Hence, object definition is related

to the meaning or purpose the object has in an existing life situation. Machine identification of targets requires machine interpretation of targets. In order to relate to the information the machine will provide, some semantic or subjective elements have to be taken into account in the data processing.

This work will be an attempt to provide a systematic theory for target identification. The theory will address itself to basic questions such as: What is a target? How can I define a target as an object or as the concept of an object within a mathematical context? To the author's best knowledge, this work is a new approach, which may bear fruit in many other fields of application.

If one wishes to identify targets, one has to have at one's disposal a system of knowledge. And in order to have a mathematical approach, one has to develop first a mathematical theory of cognitive systems. The present scene shows a proliferation of cognitive sciences. This is no doubt due to the introduction of computers which propels artificial knowledge into society at an unprecedented rate. With all this activity, it is rather surprising that no corresponding theoretical activity on knowledge based systems is visible today. Most approaches are of a heuristic nature, resembling the state of physical science in an ancient period.

Perhaps the difficulty lies with a basic stumbling block alluded to above: one does not know how to present subjective elements within a consistent mathematical framework. Most common logic is based on a binary notion of truth: something is true or it is false. However in real life, one often encounters questions to which the answer must be: "I don't know". Basically, subjectivity is hidden within this phrase, because it refers to a state of personal knowledge. The addition of "I don't know" as part of the system of knowledge leads to a simple system of (modal) logic which includes objective as well as

subjective elements as a necessary and essential requirement. An important result of this part of the work will be the notion of: "Truth within a finite system of knowledge". The next task will be to find a precise mathematical representation for the knowledge operations defined previously. This part rounds off the theory for cognitive state-operations.

A further crucial step is looking for processes which lead to formation of knowledge states. Two existing technologies are useful as prime models for these processes. The two technologies are associated with radar and quantum mechanics. At first sight, these fields seem to have little in common with each other and with knowledge in general. However from a knowledge-based point of view, a radar system can be viewed upon as a sophisticated model and extension of human sensors to look at and interpret the world, whereas quantum mechanics provides a refined theory of measurement and interaction with the world of elementary particles. In both cases, what is emphasized is the identification of objects placed within a given real-world environment based on their characteristic parameters and their dynamics.

The borrowing of tools from well-established technologies obviously has many mathematical and practical advantages. Primarily, it opens up new ways for looking at things based upon known principles. Hence it extends conceptual horizons with the least amount of difficulty and with the most possible gain to be achieved.

The following sections lead systematically to this new world of knowledge, the subject matter being that of knowledge itself and how it relates to target identification.

SECTION II

ELEMENTARY THEORY OF KNOWLEDGE

Setting the Stage

The ability to know reaches into almost every conceivable human enterprise: ordinary observation, education, development of skills, science and engineering, medicine, practice of law, government, mathematical skills, written and verbal expression of thought, psychology, etc. The list could be extended almost indefinitely, since there seems to be no boundary to the application of knowledge. Any person or other organism which possesses knowledge of some kind is said to be a cognitive system.

Although everyone has an intuitive feeling about what knowledge is and does, as soon as one tries to express this feeling in precise terms, a baffling variety of possibilities arises due to the variety of subjects to which knowledge applies.

The task of defining knowledge can be made somewhat simpler by recognizing that knowledge is somehow related to a statement of fact. Somehow knowledge relates to statements of truth about facts or events. This relationship is commonly used to test knowledge.

Suppose a teacher has taught a course and now wants to test the students' knowledge on the subject. A simple and efficient way would be to expose the students to a multiple choice type of quiz. Each sentence expresses a statement of fact related to the course subject matter, which has three mutually exclusive entries: "true", "false", or "I don't know". By this method, the students' knowledge on the subject could be tested, although some guessing might cloud the accuracy of the test.

In order to eliminate the influence of guessing on the test and provide a more accurate measure of knowledge, a backup procedure could be initiated. The backup procedure would require each student to prove the accuracy of the assertion for each entry made on the questionnaire.

The backup procedure provides a foolproof test for knowledge at the expense of considerably complicating the originally conceived simple multiple-choice quiz. There are several reasons that the above sketched foolproof procedure for testing knowledge is, in most instances, too complicated to be carried out in practice. While the original test consisted of statements whose validity can be asserted in an objective, neutral manner (i.e., a computer can provide for each student his own test score), the backup procedure would require a skilled staff to evaluate the answers. Each student will have, perhaps subjective, arguments for reaching decisions, and the validity of these arguments have to be verified by skilled personnel.

There may even be disagreement among staff members about the accuracy of the students' proof, and this matter could possibly be resolved only by bringing in a higher authority or expert. Hence, every backup procedure might require its own backup. In fact, the process described here is the way knowledge actually progresses in scientific investigations. At a certain stage of knowledge, a set of statements related to a scientific inquiry are considered by the majority of experts to be true. Then, at a later stage, some experiments show inconsistencies with this established theory. The established theory then has to be modified to incorporate the newly observed facts, and the new knowledge is now accepted as the truth by a majority of experts, until, at some later stage, new facts require theory modification.

It follows that knowledge of a subject matter requires that its definition has built in the possibility of backing up or

regressing. A complete or adequate definition of knowledge thus requires the possibility for an infinite regression^{*}; this property is a basic and essential requirement of all systems of knowledge.

Formal approach to cognitive systems

The above sketched informal approach to cognitive systems is useful to present a formal theory. The quiz situation is particularly tractable to formalization, although later on the range of applicability of the theory will be considerably extended to include almost any kind of event situation.

The quiz consists of input sentences of a propositional kind called x . The output consists of sentences y , mutually exclusive in three categories, which are of much simpler kind ("I" indicates the student):

- | | | |
|---------|--|--------------|
| y_1 : | "I can show x is true" | (true) |
| y_2 : | "I can show x is false" | (false) |
| y_3 : | "I cannot show x is true or x is false | (don't know) |

On the right-hand side are denoted the three possible entries, in shorthand notation, as they would appear on the questionnaire.

First sentence y_1 is formalized. Since x is a statement and y_1 is a statement, a system of knowledge A , which transforms the x statement into the y_1 statement is introduced.

^{*}In practice, however, the regression can only be finite, and this fact has important consequences for the development of cognition in science.

Let $y_1 = Ax$: "I (A) can show x is true",

where (A) in the sentence on the right is included to indicate that the knowledge A belongs to the person who makes the statement.

Since A in the above definition has the mathematical character of a transformation, and later on different speakers with knowledge A_1 and A_2 interacting (communicating) with each other will be encountered, it is required that if A_1z is true, where $z = A_2x$ is a statement of the type y_1 spoken by A_2 , then $A_1z = A_1(A_2x) = (A_1A_2)x$. That is, the combination of statements may be considered as a combined system of knowledge A_1A_2 operating on the sentence x.

This system of knowledge can be expressed as:

A_1A_2 x: "I (A_1) can show that: 'I (A_2) can show x is true';
is true"

Several comments can be made at this point. Suppose the speaker with knowledge A_1 can show that x is true such that A_1x is valid. Then the validity of A_2x can be established, without having recourse to the method by which the speaker with knowledge A_2 (speaker A_2) intends to prove that x is true, hence, A_1A_2x is a valid statement.

On the other hand, if speaker A_1 cannot prove for himself that x is true, then speaker A_1 can only verify speaker A_2 's statement: A_2x , if speaker A_1 finds out the method by which speaker A_2 intends to prove his contention. In the latter case there is the back-up requirement for speaker A_2 to present his proof. After hearing the proof, speaker A_1 , having learned something, can now give consent by uttering the statement: $A_1(A_2x)$.

The simple statement: $y = Ax$: "I can show that x is true" cannot be of much interest, unless speaker with knowledge A (speaker A), presents the method to show Ax is true. In fact, this is what the following statement implies.

$$z = Ay = A (Ax) = A Ax:$$

"I can show that: 'I can show x is true', is true."

The last sentence contains, in essence, the willingness of speaker A, after uttering: Ax , to back up the contention. Formally, the essential requirement of a cognitive system is now presented. The statement: Ax implies: $A Ax$, or $Ax \rightarrow A Ax$, and since already $A Ax \rightarrow Ax$, $Ax = A Ax$, for all statements x for which Ax is true, from which follows formally:

$$A A = A$$

The rule $A A = A$ is the basic rule for all cognitive systems considered here.* From it follows easily the infinite regress requirement, which is characteristic of all such knowledge:

$$A = A (A) = A (A A) = A A A = A A A A \dots$$

Notice that the associative rule which is essential to the formal development of cognitive operations:

$$A (A (A (A))) = (A A) (A A) = A (A A A) = \dots = A A A A$$

This rule applies in general, i.e., $A_1 (A_2 A_3) = A_1 A_2 A_3$, etc.

In most practical cases, one is rarely required to go beyond the first regression or back-up $A = A A$, in order to give a

*For variations on this theme, see Reference 1.

convincing proof of statement Ax . For example, in Euclidian geometry, a formula or statement x can be proven by stating $y = Ax$: "I can show that formula x is true", if a backup statement or proof: Ay which reduces the theorem x to the axioms of Euclidian geometry can be produced. The axioms are considered true by definition and no further backup is required, nor would it be of any further interest.

Such a mathematical cognitive system of proof is called closed, for obvious reasons, whereas knowledge in physics is always approximate. Always there is a possibility for backup and improvement. Such a physical cognitive system is then called open or open-ended.

In mathematics, if an operator satisfies the rule $A = A A = A^2$, the operator is called idempotent, or a projection-operator. Because in our example A serves to transform a sentence x into a sentence $y = Ax$, it is natural to look for a representation of A in terms of customary mathematical representations of linear transformations. These are $n \times n$ ordered sets of numbers called matrices (n is called the order of the matrix) which have the associative rules for multiplication $A_1(A_2A_3) = (A_1A_2)A_3 = A_1 A_2 A_3$ as required by our system. The idempotent transformations are naturally represented by a very special type of matrix representation called dyadics or outer products of vectors. All these topics are developed systematically in the following sections.

Truth and Negation Operators

In this section an important special case of cognitive system is developed. In a sense, it is an absolute system of knowledge, in contrast to the system of finite knowledge as discussed before. Recall the definition of A applied to the sentence x :

Ax : "I can show that x is true."

In order that the statement Ax be valid, the speaker A must be willing, upon request, to make back-up statements A (Ax), A (A (Ax)), . . . etc., as discussed earlier. All this was implied by Ax, since $A = A \ A = A \ A \ A = \dots$ etc.

Once Ax is valid, it must be that "x is true" also is a valid statement. The following two notations are introduced:

Tx: "x is true"

Nx: "x is not true" (or "x is false")

First, notice the difference between Ax and Tx. The statement Ax: "I can show x is true" requires some kind of proof which depends on the knowledge A, the speaker, possesses. In other words, speaker A has a subjectively (or privately) oriented procedure for proving Tx: 'x is true', whereas Tx is a statement of fact which obtains an 'absolute' character, once 'x is true' has been proven beyond any doubt by at least one cognitive system A. Also, the validity of Ax cannot be shown for every proposition x which happens to be true, since a cognitive system A is, by its very nature, limited. Speakers with knowledge A will not be able to pass judgement on every true statement x. Recall the quiz option y_3 :

y_3 : "I cannot show that x is true, or that x is false."

of the preceding discussion, which illustrates this point.

An obvious example of the limited nature of personal knowledge is for a speaker A to be confronted with a true statement x in an unfamiliar foreign language.

This recalls the further possibility that x is a "true" sentence in some extinct language, not presently known, or

understood. Can x be labelled "true" in such a case? The significance of the absolute statement Tx : " x is true" has to be adjusted to disallow such cases, since obviously, there is no method by which " x is true" could ever be proven valid here.

Another definition, instead of Tx , seems to suggest itself:

T^1x : "One can show x is true"

with the further understanding that "one" in this sentence must be some authority with a cognitive system A_0 who is part of a community of scholars with knowledge A_i and who can show A_0x to be valid. For that case, the statement Tx : " x is true" is an "absolute" truth, which has validity only with reference to the class A_i of cognitive systems considered. This actually is the typical situation for open systems in the physical sciences which were discussed earlier.

Henceforth, Tx : " x is true" will be considered to have meaning only in the latter sense, i.e., Tx has meaning only within a class A_i of cognitive systems to whom a proof of validity must be available and accessible.

Returning to our formal discussion of Ax , Tx and Nx , it was found: Ax implies Tx ; $Ax \rightarrow Tx$, but the converse obviously is not the case, since A has finite knowledge.

Next, putting $y = Nx$: " x is not true" (or " x is false") then $Ay = A(Nx)$: "I can show ' x is false', is true" or simply:

$y_2 = A_Nx$: "I can show x is not true" = "I can show x is false"

where a shorthand notation A_N is used* to denote the operator working on x , and the second option of the quiz entries discussed

* Later it will be shown that A_N and AN are not exactly the same operators.


previously is recognized. One consequence of the above statement is that from $A_N x$ follows Nx or, formally: $A_N(x) \rightarrow Nx$.

Consider the last result as a special case of the previous one $Ay \rightarrow Ty$, where $y = Nx$. From this observation, $T Nx = Nx$ for all meaningful x and hence $T N = N$ is an operator identity. Since $Nx = "x \text{ is false}" = "x \text{ is not true}" = N Tx$ applies, also, for all meaningful x , $N T = N$ is another operator identity. Similarly, $T T = T$ can easily be verified as an operator identity.

The only remaining multiplication rule to prove is: $N N = T$. This statement reflects the fact that for every sentence x which happens to be true: Tx : " x is true", a sentence $y = Nx$ can be constructed which is false; therefore, Ny : " y is false", is valid. Hence the last statement Ny is just another way of stating that Tx is valid: $Tx = Ny = N Nx$ for every allowable x , from which follows $N N = T$.

From the above discussion, a general principle was found; for every allowable or meaningful statement (proposition*) x , either Tx : " x is true", or Nx : " x is false" is a valid statement. The fact that for every x statement a corresponding $y = Nx$ (conjugate) statement may be constructed has been established. All the above results may be summarized in the "multiplication diagram" shown in Table 1, which applies to the so-called "absolute" system T :

TABLE 1. MULTIPLICATION DIAGRAM FOR ABSOLUTE SYSTEM T

	T	N
T	T	N
N	N	T

*Only a propositional statement x which can be used in the quiz situation discussed earlier is considered.

The diagram should be read as follows: Start with an entry from the left-hand column and multiply by following the arrow with one of the upper row entries of the diagram. The answer is to be found in the corresponding square of the block enclosed by the entries.

Notice at once that the (T, N) operators form a commutative group with identity T . Furthermore, since $T T = T$, the operator T satisfies the fundamental idempotent law for a cognitive system, and thus T may be considered as a limiting form, or an absolute system, of all the knowledge contained within a set of cognitive systems A_i with which it is associated. Compare this with our previous discussion concerning set A_i .

Note that N does not behave as a cognitive operator ($N N \neq N!$). In order to underscore these points, the following four definitions relating A and N are presented:

Definitions:

$y_1 = A x:$	"I can show x is true"
$y_2 = A_N x:$	"I can show x is false"
$y_{31} = N_A x:$	"I cannot show that x is true"
$y_{32} = T_A x:$	"I cannot show that x is false"

The statement y_3 , as previously defined, is a combination of the last two statements, y_{31} and y_{32} . If both $N_A x$ and $T_A x$ are valid statements, then the cognitive system A is ignorant of x ; the "I don't know" entry on the quiz applies. On the other hand, $N_A x$ could also signify that $N x$: " x is false" applies and similarly $T_A x$ could indicate that $T x$: " x is true" is the case. The

statements y_{31} and y_{32} are interesting, since they express a certain ambiguity. What precisely is said when someone says: "I cannot show that --."? On the one hand, the statement is perfectly sensible, it has a well-determined significance. On the other hand, there is a certain vagueness about what this significance signifies.

The statement $T_A x$: "I cannot show that x is false" shows that the speaker lacks an ability, i.e., the ability to prove that x is false. The speaker might be not completely ignorant regarding x , since the sentence could imply that the speaker can show that x is true, but leaves this possibility open. The speaker creates a certain puzzlement on the side of the receiver as to what the intentions really are. In many typical cases of expression of human, finite knowledge, this ambiguity is present and indeed plays an important part in communication.

Another interesting consequence of this is: suppose statement x is true; hence, Tx is valid. Then if someone says: $T_A x$: "I cannot show x is false", a valid statement is made, even though the speaker might not even understand the significance of x ! One is thus forced to conclude that the set of sentences x for which x is true, which designated $\mu(T)$, is a subset of the set $\mu(T_A)$ for which $T_A x$ applies!

The same argument applies to the conjugate case: $\mu(N) \subset \mu(N_A)$, where \subset is the sign of inclusion.

Thus, an important inclusion scheme connecting the various sets of statements is discovered:

$$\mu(A) \subset \mu(T) \subset \mu(T_A)$$

and
$$\mu(A_N) \subset \mu(N) \subset \mu(N_A)$$

Notice the curious fact that the "truth set", $\mu(T)$, lies between the sets $\mu(A)$, of sentences x for which Ax is valid, and

the set $\mu(T_A)$. The set $\mu(T_A)$ is clearly larger than $\mu(T)$! The reason for this is that $\mu(T_A)$ could contain sentences x which are false (!) and hence Nx applies. This is easily verified from the statement: T_Ax : "I cannot show x is false", i.e., it could be that x is false but I can't show this. All these conclusions are somewhat surprising results which follow from the ambiguous statements: T_Ax and N_Ax .

Later the operators T_A and N_A will be found to indeed have a basic significance. They express "truth" and "negation" within or for the associated finite cognitive system A ! The results of this section can best be summarized by the diagram in Figure 1.

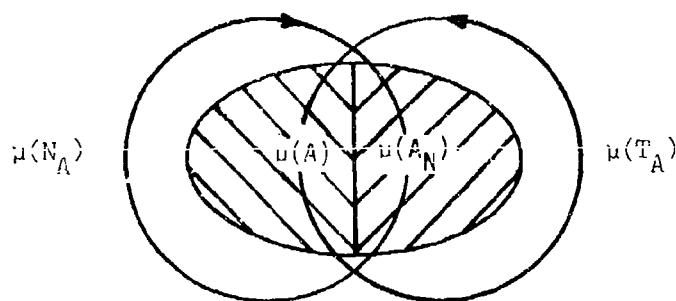


Figure 1. Relationships Between Sets of Sentences

The diagram depicts the various sets of sentences x : $\mu(A)$, $\mu(A_N)$, $\mu(T_A)$ and $\mu(N_A)$ for which, respectively, Ax , A_Nx , T_Ax , and N_Ax are valid statements. Notice the symmetry between sets $\mu(A)$ and $\mu(A_N)$ and similarly between $\mu(T_A)$ and $\mu(N_A)$. The set $\mu(T_A)$ includes $\mu(A)$, but not $\mu(A_N)$, and incorporates also

all of the "unknown" area outside of $\mu(A)$ and $\mu(A_N)$. Similarly, $\mu(N_A)$ includes $\mu(A_N)$, but not $\mu(A)$, and it includes also all of the unknown x outside of the "I can show --" group $\mu(A)$ and $\mu(A_N)$.

The Notions of Truth and Falsity are not Symmetric

In conventional logic, the notions of truth and falsity are considered as symmetric operations, similar to the binary position of a switch. A binary switch has positions which are either "on" or "off" and similarly some proposition x can be "true", in which case Tx : "x is true" applies, or "false", in which case Nx : "x is false", applies. The idea of symmetry arises from the fact that for every sentence which is true an equivalent sentence can be constructed which states the same thing but which is false.

Of a red ball it can be said: "This ball is red," which for this case is a true statement, but it can also be said: "This ball is not red", which is then a false statement. If the fact that the second statement is known to be false is added, this can be expressed by saying: "This ball is not, not red," which is now equivalent to: "This ball is red."

This property of symmetry and the analog with the binary switching arrangement has led to the erroneous conclusion that the statements Tx and Nx , and hence the associated operators T and N by themselves, also must be symmetric in nature (i.e., that they can be represented by +1 and -1 as in a binary operation).

Consider the two statements:

Tx : "This ball is red"

and

Nx : "This ball is not red"

Both statements may be independent of each other, i.e., two different balls may be under consideration. Clearly, the statement:

Tx: "This ball is red"

contains vastly more information than the second statement:

Nx: "This ball is not red".

The Tx statement makes a factual commitment, whereas the statement Nx still leaves us in the dark as to the actual nature of the color of the ball. All that is known in the last case is that the ball is not red, but it might still be blue, green, or yellow. Hence the N-statement has a very limited commitment range!

This may be one of the reasons why it is so much easier, in real life and in general, to say: No! It is easier because saying "no" does not commit one as strongly as saying "yes"! This property is reflected in cognition because $T = T T$ satisfies the cognition requirement, but $N \neq N N = T$ does not. From this follows that the T and N operators cannot and should not be represented by a binary plus and minus operation scheme, since they are not symmetric operations.

In the following section, this distinction will be made even more apparent through an algebraic argument, which leads naturally to a representation of the T and N operators.

Decomposition of Truth and Negation Operators

The multiplication diagram, shown in the previous section, gave basic rules for adjoining T and N operators. Closer observation of the diagram shows, however, that a simpler and perhaps more basic system must exist. In mathematics it is

customary to introduce linear transformations to represent an operator. This relates to a choice of suitable basis or reference system which gives the representation of the operator a most simple and suitable form. One favorite trick is to choose a transformation of the operators such that for the transformed case the multiplication diagram becomes diagonal. This is interpreted such that all terms in the diagram, except on the main diagonal, will be zero.

It is easy to construct such a transformation for the T and N operators. The following transformation is proposed:

Definition:

$$T = U + V$$

$$N = U - V$$

The U and V operators now are the transformed operators which replace T and N. Instead of the T, N multiplication diagram, the following multiplication rules between U and V are proposed:

TABLE 2. MULTIPLICATION DIAGRAM FOR THE TRUTH SYSTEM
UTILIZING U AND V

Γ^+	U	V
U	U	\emptyset
V	\emptyset	V

It is observed that, indeed, the U and V table is diagonal since \emptyset serves to indicate "zero", i.e., $U V = V U = \emptyset$. In mathematics, if multiplication between two operators gives "zero", it indicates that they are independent of each other, and

they may be called orthogonal, after the terminology in vector algebra. Hence, U and V are independent operators. Furthermore, each satisfies the basic rule for cognitive systems: $U U = U$ and $V V = V$. All that has to be done in order to verify that indeed T and N satisfy the ordinary relationships is to substitute and apply the new multiplication rules. For example: $T N = (U + V)(U - V) = U U - V V = U - V = N$. The reader can verify that $T T = T$, $N N = N$ and $N T = N$ by using a similar procedure.

A very important and very basic conceptual point is now considered. Consider the relationships:

$$T = U + V$$

$$N = U - V$$

Suddenly addition and subtraction of operators occur in addition to multiplication! The logical "and" will be designated to the + (plus) sign and something like "and lack of" will be assigned to the - (minus) sign.

Meaning to Ux and Vx will be assigned such that

$$Tx = Ux + Vx: \text{ "x is true"}$$

$$Nx = Ux - Vx: \text{ "x is not true"}$$

remain valid.

The next step consists in trying to discover a meaning for the symbols Ux and Vx . The simplest seems to be the identification of Vx , since $+ Vx$ suggest "truth" for x of some type and $-Vx$ suggest "lack of truth" for x of some type. These interpretations approach very closely to the meaning of "x is true" and "x is false".

In order to make a precise distinction, the following definitions are proposed.

Definitions:

- + Vx: "x is objectively true"
- Vx: "x is objectively false"

In other words, +Vx means the fact that x is true can be objectively asserted (someone has an accessible proof for x). The latter phrase agrees with our previous discussion regarding the significance of "x is true" with relationship to a set of cognitive systems A_i .

Next, Ux needs to be given meaning. Notice that Ux appears both as part of Tx and Nx and U is independent of V, such that Ux is independent of "the objective truth of x." Hence, Ux asserts something about x which is independent of its truth content. But surely, x must exist, before a truth verdict can be given to it! The following definition is proposed:

Definition:

Ux: "x exists"

Sensitive philosophical grounds are now involved. What does it mean for a thing to exist? Perhaps it can be agreed upon that a rock exists, but does a number exist, does a phrase exist, and what about: x: "The present emperor of France wears a red hat"; does that statement exist? Interesting philosophical literature exists on this subject*. At the moment, all subtleties will be avoided and a naive viewpoint will be taken. As the theory progresses, refinements and modifications of our interpretation will be introduced as it seems appropriate and necessary.

* Reference (2).

The naive interpretation of Ux : "x exists" is simply that x must be available or accessible in some way and that x has an appropriate meaning or significance as a well-defined entity in the form of a propositional sentence. These all seem logically necessary and perhaps sufficient conditions which any reasonable x must have, in order that an objective label Vx of "true" or "false" can be attached to it.

The significance of Ux : "x exists" could be pushed a bit further if x is a statement about an object. Then "x exists" can be interpreted to signify a reference to the object itself, and it can be inferred from "x exists" that "the object exists." On the other hand, if "the object exists" then the sentence x referring to the object must have meaning and hence, "x exists." From this follows that for this case: "x exists" and "the object exists" are identical statements.

For example, the true-false relationship is often compared to a binary switching arrangement. Hence, if x is the sentence: "the switch is on," then Tx : "x is true" applies if indeed: "the switch is on." If "the switch is off" then Nx : "x is false" would apply. In this particular case, what is "constant" in the sentence x is the phrase "the switch", and Ux : "x exists" can only be the case if "the switch exists" is a valid statement, in the sense of: without "switch" there would be nothing to switch "on" or "off," and no statement x could refer to it! For this case, it would be appropriate to have for x: "the switch is on," the following interpretations where it is assumed Tx is valid:

Tx : "the switch is on."

Nx : "the switch is off."

Vx : "one can verify that the switch is on."

Ux : "there is a switch."

The usual method in dealing with the switch analogy above is to assign integers ± 1 to "on" and -1 to "off" positions. (This is common practice in computer algorithms, for example.) The fact that there has to be a switch in order to make the algorithm work is usually taken for granted and hence, the Ux aspect of the situation is usually ignored.

This points out an important fact of the decomposition $T = U + V$; V refers primarily to objectively assertable truth, while U refers primarily to subjectively assertable existence or being. That is, the fact that the switch is "there" has to be asserted by a subject who actually is confronted with the situation of verifying the position of the switch.

All that seems implied by the statement Ux: "x exists" and usually, in the objective mode of dealing with a subject matter, this co-called "existential" mode is completely ignored. This observation will be a recurrent theme in subsequent investigations.

Language as an Objectification Process

Since almost everything that is known is expressed through the use of verbal or written expression of language, it seems appropriate to study the language process in some detail and analyze its function in communication. Let us start with a simple example: I hold a cup in my hand and someone starts a conversation about its content: "You like your coffee black?" or "Is it not too hot?" These questions refer to common knowledge about a cup in general and its useful function in life. The dictionary definition of "cup": "A small, open container for beverages, usually bowl shaped and with a handle," suffices to provide an objective understanding. Anyone wanting to refresh his memory of cups can consult the dictionary, as I just did.

But how does the dictionary definition relate to the real cup I have in front of me? I notice that my cup is more

cylindrical than bowl shaped. It also has flower patterns on its side which the dictionary does not mention.

A little reflection will convince us that what the dictionary does is to provide us with the most essential type of information that is necessary to define the cup in general, but it does not even claim to attempt to describe my cup.

If I were a poet, I could attempt to capture the essence of my cup through language by using poetic imagery. Even that may fall short of the total awareness I have by holding this cup up to my attention at this moment. Clearly my experience of the cup event is a subjective one, whereas the dictionary definition provides objective knowledge, accessible to all. In communication one generally uses objective (common) knowledge to describe real events, in contrast to the real-world experience of the event itself. If we both witness the same event, there is no need to communicate this, aside from the above mentioned difficulties one encounters if one tries to do this in detail.

Hence the use of language may be considered as a mapping of the real-world event situation onto an objective framework. By doing so, something has happened to the original event: It has lost the uniqueness of its being there (called Ux in the theory). The dictionary is a mapping of all common real life knowledge of objects or events onto the organization of words and pages and paper which is the dictionary. The dictionary gives us an objectified definition of our cup; it cannot present us with the real cup. That this must be so is due simply to the fact that the dictionary is available to all.

The "objectifying" property of common language has serious consequences for our processes of understanding. By continually "abstracting" events we are in danger of losing track of understanding the significance of a real-world event. The real-world event is related to my presence as an observer to witness the event. Many philosophers of East and West have pondered

these questions and have come to the conclusion that the real-world event situation is something of a mystery. This is but another way of saying that objectified knowledge and understanding cannot grasp the significance of the real-event case (we can see how this applies in probability theory). We could quote a large volume of literature on this subject, from Zen Buddhism, the ancient Mystics, Hegel, Kant, Kierkegaard, Marcel (The Mystery of Being, Vols I and II), Husserl, Heidegger and a host of others. The trouble with all this activity, obviously is, that it is itself expressed in language! How can we express in language something which cannot be expressed in language?

Such logical puzzles have plagued much of modern philosophy, to the extent that some (the logical positivists and some linguists) will deny any meaningful discussion of what we would call "reality"! If the whole world becomes objectified, then there is no place for reality because an objectified world is a possible world and cannot be the real world we live in.

Is there a fool-proof method for knowing that a real world does exist? The answer is yes, but we will never succeed by using purely rational arguments! Then, why not break the consistency of logical discourse! The way reality is usually impressed most vividly upon us is if we make a mistake. We lose a key only to find it back the next day on the couch. The event takes place without a logically consistent knowledge of the whereabouts of the key. Finding the key impresses upon us and restores to us our confidence about the continuity of the real world. This is how reality is "found" and rediscovered. Notice that the process above works because no linguistic modes of expression are used to achieve the desired result; only actions are necessary.

Objective Truth and Being

There is yet another way to interpret the decomposition $T = U + V$ and $N = U - V$ which is perhaps more "technical" and hence less "mystical" and, thus, easier to understand. We commented on the use of language and the difficulties one encounters because of the natural objectifying tendency language possesses. Language must express itself through written or spoken symbols which by their very nature and purpose must be understood by all (who engage in the communication) and hence must be objective in character. On the other hand, what is experienced is private and singular. By trying to communicate to others what I have just experienced I must try to "objectify" this experience. In mathematical terminology, we call this process a mapping of event x onto an objective framework, expressed by Vx : " x is objectively true". We can think of statements like: "This is a cup", "The cup is cylindrical", "The cup has flower patterns on its side", "This cup has no handle", etc. In fact, almost everything I have experienced can thus be brought into an objective framework.

Why do we call this process "objective"? Simply because all the statements above itself are symbolic representations of something real. A "cup" is not a real cup; it is a word consisting of three letter symbols, which is the projection of the real-world cup: x onto the objective representation: Vx : "cup". By the way, this is why $V = V V$ is called a projection operator. Similarly $UU = U$ is a projection operator. But we notice a curious difference if we compare the two types of projections:

Ux : " x exists"

Vx : " x is objectively true"

We notice that while Vx may be used to express numerous properties of a real object such as a cup, the statement Ux : "x exists" seems singularly impoverished by comparison! Ux seems only to refer to one property of a real-world object x : its unique existence!

Another way of saying what we just have found is that almost everything of a real-world event x , related to its truth content, can be objectified, the singular exception being its factual existence!* A little reflection will convince us that this result is indeed very plausible. If I consider my own situation as a real-world event, then everything around me can be "objectified" or generalized. I even can look in a mirror and see myself as others may see me. I am "a person" just as I see other persons. This is the objectified picture I can develop for myself. However there is one quality missing in this picture: I am this person which I see in the mirror. The fact of my being myself cannot be generalized, because I cannot transfer my being to others. Obviously I can generalize the concept of being to others, but, in fact, I am stuck with my own being, with what and who I am and this fact cannot be generalized!

This is really what: Ux : "x exists" tries to express in mathematical terms. It refers to the individual, private, real-world fact of being of event x . The equation $Tx = Ux + Vx$ hence expresses a decomposition or projection of x into two orthogonal spaces: One is the "truth - objectifying" space V , whereas U contains only one singular aspect of x : its real-world existence.

One notices how simply the algebraic expression describes the process compared to the almost desperate and frantic verbal

*Later on we will find how this rule can be expanded for a finite system A , to include all objective and all subjective experience.

expressions of the philosophers! The verbal effort, by using ordinary language, is almost incomprehensible in this case, because language itself has the objectifying quality built into it, and the U-frame, by definition, escapes any attempt at objectification. It is no wonder that in a society, which is raised on objective-only concerns, there will be a tendency towards neglect of subjective concerns. The present work on cognition is an attempt to restore some of the imbalance, by presenting a self-consistent algebraic scheme which incorporates subjective as well as objective elements of knowledge.

Decomposition and Significance of Cognitive Operator

In a previous section, the "truth" and "negation" operators T and N were expressed in a natural way into an existential operator U and an objective truth-related operator V. It seems natural to expect that the same procedure will apply for the finite cognitive system A.

Following this suggestion, A and A_N are decomposed as follows:

$$A = S + Q$$

$$A_N = S - Q$$

Recall that A and A_N were the statements equivalent to T and N, for the finite case, where:

Ax : "I can show that x is true"

A_Nx : "I can show that x is false"

For the S and Q operators, the following rules are assumed: $S S = S$ and $Q Q = Q$ and the rule for independence: $S Q = Q S = \emptyset$. The definitions for Sx and Qx are now easily found in analogy to Ux and Vx for the absolute system. First, the cognitive system requirement $A A = A$ is verified:

$$A A = (S + Q) (S + Q) = S S + Q Q = S + Q = A$$

Now, a new rule for $A_N A_N$ can be derived:

$$A_N A_N = (S - Q) (S - Q) = S + Q = A$$

As expected, A_N is not of the cognitive system type. Also, $A A_N = A_N A = A_N$. A definition will now be presented:

Definition:

Sx : "I understand x."

This is to be interpreted as a subjective evaluation regarding the significance x has for the spectator (the subject, "I").

Now, Qx is interpreted as follows:

Definition:

Qx : "I can prove x."

Qx indicates that there is a definite objectively valid procedure by which the subject "I" intends to prove x. Hence,

Ax : "I can show S is true" =

"I understand x" and "I can prove x."

If $A = S + Q$ indicates the cognitive system, then S is called its subjective support and Q (for quest of truth) stands for objective truth in the finite model.

The decomposition of the cognitive system A into natural subjective and objective components provides greater insight into the structure of the system itself. The subjective part S provides the necessary support to the system of knowledge, by which significance and meaning of the issue at hand are evaluated in terms of individual interaction.

If someone yells: "Watch out for the car!", the significance of the issue to survival of the individual is clear, and evasive action is essential for preserving it. The support mechanism S not only provides significance to the self based upon evaluation of primary data relating to the event, it also organizes a course of action to be taken if action is deemed necessary. Q is the operation which executes the plan of action.

Take, for example, a mathematical problem in geometry. Let Ax: "I can show that 'theorem y' is true", where x = "the theorem y" = "the Pythagorean theorem." Now, Sx: "I understand x" has to be interpreted as follows: First of all, the speaker grasps the significance of the theorem and knows how the theorem relates to other facts in geometry, i.e., lines, triangles, right angles, etc. But "understanding" in this context requires more. The subjective self S which is part of the speaker's knowledge must be able to devise some scheme of attack. The scheme may be cumbersome or elegant; this depends on the difficulty of the problem and the speaker's personal subjective skill in geometry.

Now, the objective operation Q is put to work, the program or plan of attack which S has devised is executed by Q (Q executes S's will), and the result of this action is recorded in the form of a verdict: true if the action indeed results in a proof of the theorem, in which case Ax above applies.

The given description fits the proposed designation of Sx and Qx:

Sx: "I understand x"

and Qx: "I can prove x"

As it is described, the role of S is that of a superior or supervisor, whereas Q's role is that of an executor. Q executes the plan of action or the force of the will which S imposes.

The analogy with computers is evident: The S function is carried out by the scientist-engineer team, the Q part is

represented by the programmer-computer unit which executes the program. In this instance, the scientist's part of the S-function is to present a meaningful description of the task at hand to suggest a course of action. The "engineer" part translates the scientist's ideas into a specific task. The programmer translates this task into computer language, while the machine provides the computed result or output.

The fact that the scientist-engineer's task is called subjective should not be interpreted (as is often done) that this action is arbitrary or frivolous in some way, and that the action is not guided by sound objectively assertible principles and facts. The action is implied to be subjective because there is some personal involvement necessary, first to present the case as a meaningful task within the context of the "life-situation" at hand, and secondly, to produce, out of the maze of possibilities, a course of action which will resolve the problem created by the issue.

Notice that the S, Q distinction in the analogy cannot be drawn sharply, since the programmer's subjective, independent judgement will be used to decide how best to implement the computer task. All this is quite in agreement with the theory and is to be expected, since the programmer, as a cognitive system, possesses subjective as well as objective skills. The computer, being a machine, can be said to have only pure, objectively related Q-type functions.

These results show that with the few mathematical developments thus far attained, many interesting, sometimes controversial, conceptual issues have already been examined. The above sketched man-machine distinction is one such issue. The theory indicates that a true cognitive system must possess a subjective as well as objective part, while a machine, at least in the present stage of development, has only objective functions.

It is the task of the subjective "self" of a person to provide meaning and significance to issues. Most present scientifically oriented doctrines are unable to cope adequately with subjective concepts largely because of lack of a scientific framework to discuss and approach these subjects.

Science has been very successful with the so-called "objective" approach, which means that scientific results are recorded objectively, i.e., in a manner accessible to anyone, and such that its truth content can be verified. All personal feelings and emotions, hints, hunches, attempts which led to failure, etc., which reflect the subjective aspect of scientific research is carefully shunned and eradicated from the final "ideally correct" record of scientific investigation. The valid idea behind this approach is that all that counts (except for historical anecdotes) is the objective record of an event which now becomes part of a community of knowledge.

The only danger this attitude presents is that if too much exclusive emphasis is placed on objective presentation, the subjective element, which is an essential part of human intelligence, becomes the stepchild who suffers from abuse, misunderstanding, lack of nourishment, and loving care.

A slightly different viewpoint can throw additional light upon the distinction between "subjective" and "objective" attitudes. The objective aspect of knowledge deals primarily with "possible" situations, or "possible worlds." It is in the nature of scientific presentation to deal with issues as members of a class of similar issues. For example, with a rock in hand, one can make the observation "this rock is hard." Translated into objective terminology "hardness" of rocks can be made a precisely measurable scientific quantity. By doing so, all rocks can be classified according to the number of units of hardness they possess. All this activity tends to draw away attention from the fact which the speaker was originally confronted with: This rock "feels" hard.

The "objectified" rock now becomes a member of a class of rocks, each of which has a property called "hardness." The actual rock has lost, as it were, its uniqueness, which characterized the event of holding the rock in one's hand, instead it has become a member of a class of rocks. Scientific presentation is solely interested in this type of class-membership. An actual, factual object, becomes a possible object, only to be dealt with as an accidental member of a class of possible similar objects.

The reason for this is that for a possible world to become an actual world, a subject has to be present which evaluates and decides to act upon the actual world. In science, such a subject is called an observer, investigator, or scientist, who delivers the support function S which enables him to evaluate and perform experiments on actual events.

The neglect of the subjective aspect of knowledge of human beings can lead to serious failures in modeling structures which involve human, and generally biological, activities such as are found in teaching, economics, psychology, medicine, religion, family life, the arts, etc.

These failures can be attributed to a failure of philosophical intent, due to the incapacity to deal adequately and in a clear, precise, concise manner, with issues relating to the subjectivity of the subject. The greatest loss is found in the failure of the scientific-objectively trained person to comprehend himself, as actually existing in a real world.

Internal Communication Processes

The art of communication consists of conveying to someone else, a willing observer or listener, the status of an event x , as one has interpreted this for himself.

Previously, it was found that this interpretation is contained in one of the four statements about x , where x in our case is a proposition^{*}:

Ax : "I can show x is true"
 A_Nx : "I can show x is false"
 T_Ax : "I cannot show x is false"
 N_Ax : "I cannot show x is true"

Each of these statements carries the message of an interpretation to follow up with a backup interpretation which result is expressed by the new statement. For example: $A(Ax)$: "I can show that, 'I can show x is true,' is true." In fact, since $A A = A$, this backup interpretation is already implied by the former statement Ax . Similarly, since $A(A_Nx) = (A A_N) x = A_Nx$, the affirmation or backup of A_Nx does not change the status of A_Nx .

But what about $A(T_Ax)$ or $A(N_Ax)$? Consider $A(T_Ax)$; T_Ax expresses a certain ambiguity of opinion regarding the status of x . If the backup, clarification, statement $A(T_Ax)$ is used, what does this signify? If the "ambiguous statement" is shown to be true, it must be that the ambiguity has been resolved, and hence, $A(T_Ax) = Ax$ applies. Later on, that this indeed is the correct interpretation will be proved. Hence, it follows for the operators:

$$A T_A = A$$

* Other logical systems such as Zeman [3] and Prior [4] define operators M_2 as "possibility" and L_2 as "necessity" where $M_2 = N L_2 N$. These can also be incorporated within the present theory.

Notice here a curious fact. The backup affirmation of $T_A x$ alters the original statement! In this case, the affirmation of $T_A x$ clarifies the originally ambiguous statement $T_A x$. It removes the ambiguity to indicate that after all Ax : "I can show x is true," is a valid statement which was already known when $T_A x$ was stated, or which validity was derived later on.

A similar analysis shows that: $A(N_A x) = A_N x$, and hence, in general:

$$A N_A = A_N$$

The above examples give cases where the backup statement perhaps modifies the original statement, but it does not contradict what has been stated before.

One can easily produce examples where such contradictions do occur! Since the identity $A_N A = A_N$ was derived previously, a clear case of contradiction is recognized. The original statement was an opinion of x ; Ax : "I can show x is true," while the next opinion (one can now no longer speak of backup here) is: $A_N (Ax)$: "I can show, 'I can show x is true,' is false!" which indicates that $A_N x$: "I can show x is false" now applies and this is in clear opposition to the first statement: Ax !

At first impression, such a "change of opinion" seems impossible, since Ax assumes an infinite regress $A A \dots Ax$, by which one can prove the first statement with any degree of accuracy and sophistication conceivable. In the vast majority of instances this will indeed be the case and a change of opinion $A_N (Ax) = A_N x$ seems very unlikely. The question here is not whether the above statement is likely, but whether it is logically possible!

All that has to be done is to change the infinite regress idea above to the realization that, in practice, there can only be finite number of steps. Now, the finite regress $A A \dots Ax$,

by which the original assertion Ax was made, does not preclude logically the possibility of change of opinion at some stage, since a finite sequence, by its very nature, must terminate. Thus, a change of opinion $A_N A A \dots Ax = A_N x$ remains open as a, perhaps infinitesimally small, possibility. The fact that there is no absolute certainty that "x is true" has important consequences for the notion of "truth in a finite system of knowledge," as shall be seen shortly.

Having thus clarified the status of $A_N A = A_N$ operators, one can now proceed with the identification of the remaining combinations of operations. The results are summarized by the multiplication diagram in Table 3.

TABLE 3. MULTIPLICATION DIAGRAM FOR THE COGNITIVE SYSTEM UTILIZING A , A_N , T_A , AND N_A

		Original first statement			
		A	A_N	T_A	N_A
Follow-up second statement	A	A	A_N	A	A_N
	A_N	A_N	A	A_N	A
	T_A	T_A	N_A	T_A	N_A
	N_A	N_A	T_A	N_A	T_A

The upper row indicates first statements while the left-hand column indicates follow-up statements applied to the first one. The resulting combined statement is contained in the corresponding box, as indicated.

The reader is invited to verify the fact that "ambiguous" follow-up statements T_A and N_A working on any of the first statements can only result in final ambiguous statements. For example, if Ax : "I can show x is true" applies as first statement, and then say; $T_A Ax$: "I cannot show that; 'I can show x is true,' is false," a definite impression of uncertainty regarding the original intention: Ax is presented with the added possibility of: "I don't understand x ," after all. Hence, Ax transforms into $T_A (Ax) = T_A x$ as shown in the box.

The multiplication rules, given here, provide the logically consistent connections between the four logical operations. The table not only shows these connections but also serves to define, in a precise manner, the logical nature of the operations.

The previous discussion has emphasized that, "truth in a finite, cognitive system," cannot be defined with absolute certainty; there is always a, perhaps infinitesimally, small chance that Tx : " x is true" has to be modified into Nx : " x is false." In other words, "truth in a system A " cannot be represented by an absolute T , independent from A and similarly "negation of truth in A " cannot be an absolute negation N , which is independent of system A .

A glance at Table 3 reveals to us, at once, what then 'truth in A ' should be. From Table 3 we easily verify the structure diagram given in Table 4.

TABLE 4. MULTIPLICATION DIAGRAM FOR THE INTERNAL TRUTH SYSTEM UTILIZING T_A AND N_A

Γ^+	T_A	N_A
T_A	T_A	N_A
N_A	N_A	T_A

This diagram shows the original (T, N) multiplication table with T replaced by T_A and N replaced by N_A !

T_A can now be identified with "truth in system A" and N_A with "negation in system A." Next it is verified that T_A and N_A indeed function as required of these intuitive notions. This follows from the fact that $A_N x$: "I can show x is not true" and: $(A N_A x)$: "I can show that: 'x is false,' is true," are identical statements. Here "x is false," which is spoken in the context of the finite system A, is to be interpreted, not as an absolute negation: Nx , but rather as $N_A x$ = "negation of x in system A."* In order to distinguish cases, henceforth $T_A x$: "x is true in A" will be written and similarly $N_A x$: "x is false in A."

The original four statements are now written as follows:

$Ax = A T_A x$:	"I can show 'x' is true" = "I can show 'x is true in A' is true."
$A_N x = A N_A x$:	"I can show 'x' is false" = "I can show 'x is false in A' is true."
$T_A x = N_A A N_A x$:	"I cannot show 'x is false in A' is true" = "x is true in A" = "x in A" is true.

Notice that formally the operation A is expressed by:

-- A .. = "I can --- show .. is true."

* This clarifies the fact that AN and A_N are not the same operators.

Where the --- position may be taken by N_A interpreted as "not" and .. is an expression of the kind shown above.

From the fact that Ax is valid, T_Ax follows and from A_Nx follows N_A or symbolically:

$$\begin{array}{lcl} A & + & T_A \\ A_N & + & N_A \end{array}$$

This result is in complete agreement with earlier results, which produced:

$$\begin{array}{lclcl} A & + & T & + & T_A \\ A_N & + & N & + & N_A \end{array}$$

What is shown here is that so-called absolute truth and negation are logically contained, but not conceptually used, as notions of "truth" and "negation" in the finite system.

What then remains of "absolute" T and N? They are to be interpreted as symbols which are operative within a community of systems of knowledge: A_i . The symbols simply indicate Tx : "x is true" or Nx : "x is false" as the "status of x", which obtains its significance only from the fact that momentarily A_ix or A_iN are valid statements.

Later, the details of the transfer of information within a community of cognitive systems (scholars) is discussed. At a certain stage of development, it can be assumed that a certain consensus will be reached, such that, for a majority of scholars, given the statement x , A_ix applies. In this case, "the truth" statement is that $T_{iA}x$: "x is true in A_i " is the case, which leaves open the finite possibility for error.

The stronger statement Tx : "x is true" which also follows logically from A_ix can under these conditions no longer be considered as an "absolute" expression of truth regarding statement x , because a finite chance may exist that the statement is in error. Tx is preferred to indicate the fact of having

obtained the status: "x is true." Later, more convincing arguments will be found to show that this indeed is a correct interpretation.

Truth in the Empirical Sciences

In the introduction it was stated that knowledge is somehow related to a statement of truth about an event x. In the last section a curious fact was discovered -- that each cognitive system carries its own notion of truth: T_A ; where $T_A x$: "x is true in A," and N_A : "x is false in A," are proper designations of truth and negation within system A.

This notion of truth has a built-in ambiguity, which seems at first rather puzzling because for every statement x for which $T_A x$: "x is true in A" is the case, there is a finite possibility that N_x : "x is false" applies! We now address ourselves to these questions.

In the first place, if A is a closed system, as in Euclidian geometry, the $A = A$ --- A will terminate at some point with a statement of truth referring to the axioms of Euclidian geometry. Hence, if x is a theorem to be proven, $Ax = A$ --- Ax will terminate with Ax: "I can show theorem 'x in A' is true," which is now reduced to a statement of "truth of x in A" based upon the accepted truth of the axioms.

The assumption is made that A does not make mistakes, i.e., that the method of proof indeed is objectively valid (this is what Ax is valid implies). By this is meant further that there is an inner consistency within system A, which is not broken by A's approach to the proof Ax. Hence, if Ax is valid, and the procedure terminates with the axioms of Euclidean geometry, then $T_A x$: "x is true in A" cannot result in ambiguity and T_x : "x is true" is a valid label that can be attached to theorem x.

At this point, axiomatic entanglements regarding consistency, etc., of a mathematical system are ignored.* The naive viewpoint that the system is closed, i.e., that proof of theorem x can be obtained without any trouble through a finite sequence of backup procedures is assumed.

It is thus illustrated that in most common procedures of proof, the question of ambiguity in the statement T_Ax : " x is true in A " has no practical significance.

But now consider an "open" system, such as occurs in the physical sciences. One can show that for such open empirical sciences, "the truth" is indeed correctly expressed by T_A , where the built-in ambiguity has important practical significance.

Consider a proposition x related to a scientific investigation. In science, such a proposition is usually called a hypothesis. The objective is to establish whether the hypothesis x is true or false; this requires an experimental investigation. After a sufficient amount of tests have been made, one may be satisfied that indeed the results of the tests indicate that the hypothesis x is true within the context of the investigations conducted.

These results can be summarized by making the following announcement: $N_A A N_Ax$: "I cannot show that 'hypothesis x ' is false." An analysis of an earlier section has shown that the above statement is equivalent to:

T_Ax : "hypothesis x is true in A "

* In axiomatic systems, such as Euclidean geometry or number theory, there exist propositions that are true or false, but cannot be proven true or false (e.g., Goedel/Church theorems). In this development, cognitive systems A_i are necessarily finite and, hence, limited in capacity to prove things. In fact, they are defined by the set of propositions which the system is capable of proving (refer to page 15).

Here x has to be considered in context A , which is the framework of experimental investigation or system of empirical science.

Now it is clear that the statement $T_A x$ expresses precisely the result of the investigation. What it expresses is that hypothesis x has been found to be true within the context of scientific investigation, i.e., $T_A x$: " x is true in A ." One can state that x is consistent within system A , but there is no absolute guarantee that ' x is true' will always remain valid!

The history of science has shown with clarity that what is considered true at one state of scientific development may have to be modified and indeed could become false at a later state of development. The "open" nature of empirical scientific development makes it mandatory that the notions of empirical truth and negation have built-in ambiguities to account for possible change of emphasis, such that there is no absolute certainty.

Hence, the case of an empirical science considered as an open system of knowledge A leads to exactly the statements of "truth in A " and "negation in A " as were developed for such a system. We can take a further step and conclude from this that the process of scientific investigation may indeed be considered a system of knowledge A .

What precisely is A when it refers to an empirical investigation? We recall that if x is a hypothesis, $Ax = Sx + Qx$ applies only if Sx : "I understand x ," and Qx : "I can prove x " are valid statements. But in what sense can I claim to "understand" the procedures of an empirical investigation? The case here is quite different from, say, a mathematical proof system. The difference is that I have no complete control of the events taking place, and I do not comprehend all relationships which connect the hypothesis x with other events of the empirical system A .

The situation is different in a theoretical investigation. Here we assume control over all possible events and can act upon them, exercising understanding. By executing judgement one can derive a proof of certain statement x . In the empirical case, it is impossible to state with full conviction: Sx : "I understand x " although this might, in fact, be the case.

This inability to secure understanding fully is, in fact, the motivation to embark upon an empirical investigation; the unknown cannot be derived, it has to be tested. This basic difference accounts for the fact that T_Ax : "I cannot show x is false" is the correct interpretation of a successfully concluded experiment, rather than Ax : "I can show x is true." In both cases what is definitely not the case is: A_Nx : "I can show x is false!" It follows that the correct statement of "truth in A ," in this case, allows for the possibility of change of opinion at some later state of development.

On the other hand, what has become of the statement Tx : " x is true"? We associate with Tx : "to indicate a status": x is true. Since $Tx = Ux + Vx$, Ux : " x exists" refers to the meaningful content of hypothesis x , whereas, Vx : " x is objectively true" is called the label, which records the status of x . These concepts are illustrated in more detail in later developments.

Internal Structure of a Cognitive System

In this section, the internal organization of a cognitive system will be developed. Our discussions so far have formulated essentially four operations: A , A_N , T_A , and N_A . We will show that these four operators indeed define the full structure of a cognitive system. We also introduced a decomposition:

$$A = S + Q$$

$$A_N = S - Q$$

which shows that A and A_N can be replaced by new operators S which we called "support in A " and Q , the "objective truth in A " function, which executes the will of subjective support S .

The advantage of the transformation above is that S and Q are independent operators: $S Q = Q S = \emptyset$ and each satisfies the general rule for cognitive systems: $S S = S$ and $Q Q = Q$. We notice that the transformation satisfies the relationships: $A A = A$, $A A_N A = A_N A = A_N$ and $A_N A_N = A$ which were shown to be properties between "truth" and "negation" operators.

Other pairs of this kind were found: (T, N) and (T_A, N_A) . For the (T, N) pair we had:

$$\begin{aligned} T &= U + V \\ N &= U - V \end{aligned}$$

where U (for universe) is an "existential" operation and V (veritas) "attaches a label" of truth or falsity. Between these operations we have:

$$U^2 = U, U V = V U = \emptyset \text{ and } V^2 = V.$$

Following these suggestions, it is natural to look for a decomposition of the third pair (T_A, N_A) as follows:

$$\begin{aligned} T_A &= L + P \\ N_A &= L - P \end{aligned}$$

where L and P satisfy the rules: $L^2 = L$, $L P = P L = \emptyset$ and $P^2 = P$. The new symbols L and P now determine the system (T_A, N_A) . Notice that L is common to T_A and N_A and, hence, represents: "I don't understand", or lack of support. L will be called the "lack" operator. It indicates insufficient understanding:

Lx : "I don't understand"

It may seem strange, at first, to associate with $T_A x$: "I cannot show x is false", $N_A x$: "I cannot show x is true", a "lack of understanding of x ", as an essential, subjective, aspect of statements of "truth in A " and "negation in A ". Some reflection on the status of elementary particle physics today vividly reinforces this interpretation.* These relationships with empirical sciences and quantum mechanics will be reviewed in later developments.

The identification of P requires rules for cross-multiplication between operators U , V , S , Q , L , and P . First notice that (U, S, L) form one set of "subjective" operators**, whereas, $(V, Q$ and $P)$ are "objective" operators, and it can be ruled without contradiction that corresponding operators in the two sets are independent of each other; $S P = P S = L Q = Q L = \emptyset$, etc.

What about operations within each set? Here a very fundamental set of rules can be derived as follows: From $T_A A = T_A$ and $A T_A = A$, we find:

$$T_A A = (L + P) (S + Q) = L S + P Q = L + P = T_A$$

$$A T_A = (S + Q) (L + P) = S L + Q P = S + Q = A$$

In working out the relationships above, we left out all products $P S$, $L Q$, $S P$ and $Q L$ which are "zero" because of orthogonality conditions.

*On page 129 of Reference 5, Feynman says: "I think I can safely say that nobody understands quantum mechanics", The M.I.T. Press, 1965.

**Strictly U is not subjective, but rather a general "existential" operator.

Now the "subjective" parts of the above result are identified, which yield: $L S = L$ and $S L = S$, for subjective operators.

For a statement x this implies: $L Sx = L (Sx) = Lx$ and $S Lx = S (Lx) = Sx$, which indicates that the subjective operator on the left always overrules the one on the right. In other words, as a general rule for the subjective operators: if S_1 is the subjective support in system A_1 , and S_2 the support function in A_2 , then if A_1 interprets the statement A_2x , it cannot possibly know the subjective evaluation A_2 has given x : S_2x , and hence, $S_1 (S_2x) = S_1x$ applies for every admissible x .

In general

$$S_1 S_2 = S_1$$

for the subjective operators. Because of these properties, this rule is called the basic rule of inaccessibility between subjective operators.

On the other hand, a similar rule for objective operators will apply. Let Q_1 and Q_2 be objective operators corresponding to systems A_1 and A_2 , then we may expect:

$$Q_1 Q_2 = Q_2$$

to be the case. This rule expresses the basic accessibility of objective knowledge.

An example will illustrate this point: If procedure $Q_2 x$ proves x is contained in a book or other record of an event x which is objectively available and accessible, then it is in principle always possible for system A_1 to acquire this knowledge: $Q_1(Q_2 x) = Q_2 x$. The only requirement for this to happen is that A_1 understands x , i.e., that S_1 also applies.

Now return to the task of identifying P. From the identification of objective components in $T_A A = T_A$ and $A T_A = A$, one finds: $P Q = P$ and $Q P = Q$. But, application of the rule of accessibility requires also $P Q = Q$ and $Q P = P$. Hence, it is deduced: $P = Q$. P is identified with "objective truth in A".

Using this result the following decomposition is obtained:

$$T_A = L + Q, \text{ and } N_A = L - Q$$

Now, since $T_A x = N_A A N_A x$: "I cannot show x is false", this phrase can be interpreted in two different ways: either as Lx : "I don't understand x", or as Ax : "I can show x is true".* The first case is clearly identified by L in $T_A = L + Q$. The second possibility is reflected by Q, since $A = S + Q$. Now Qx , the procedure by which x is proven to be true, implies a cognitive system A_1 for which $A_1 x = S_1 x + Q x$ is valid. In the context of system A which is indicated by "I" in the above statement, clearly the identification $A_1 = A$ and, hence, for this case if Qx is valid, Ax is valid as follows.

In order to incorporate all these properties with the $T_A = L + Q$ operator above, one has to interpret +, which is logically "and" as +, which is introduced as a logical symbol for "or"! Henceforth**

*The approach taken here is that of an either or alternative. In future work the ambiguity in $T_A x$ can be extended to intermediate "levels of conformation" (see references 6 and 7).

**The circled plus notation is not essential for the algebraic development and may be omitted for future work.

$$T_A = L \oplus Q$$

$$N_A = L \ominus Q$$

where \ominus is to be translated as "or not", which are the correct interpretations of T_A and N_A .

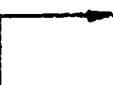
An algebra between "and" and "or" operators suggests itself; from $T_A A = T_A$ and $A T_A = A$, we derive:

$$\oplus + = \oplus \text{ and } + \oplus = +.$$

Instead of "plus", in $+$ or $+$ we could substitute "minus", $-$, such that $\ominus + = \ominus$; $- \ominus = +$, etc., following conventional rules.

For the "plus" system, the multiplication diagram is as shown in Table 5:

TABLE 5. MULTIPLICATION DIAGRAM FOR THE PLUS SYSTEM

		and	or
		+	\oplus
and	+	+	+
or	\oplus	\oplus	\oplus

with obvious relations, if a "minus" sign is introduced. The relationships above bring out the complete internal structure of a cognitive system.

Now appreciate that $T A = U + Q$ and $A T = S + V$ do not reproduce either A or T and, hence, T cannot be used as an internal or external operator describing "truth of x ". In fact, the operation $T A x = U x + Q x$ removes from $A x$ its subjective support S and substitutes the neutral, existential operation U ; $U x$: " x exists".

For the case when x is an event, it will be shown later on that $T A$ has a basic interpretation as a body B . The body function is that of a receptor and a recorder. Hence, a basic definition is adopted:

$$\text{Body} = B = T A = U + Q$$

As has been seen, in each system A , $Q x$ implies $A x$ and hence, if $B x = T A x$ is valid, then $Q x$ must be valid and $A x$ follows from this, hence, $B \rightarrow A$. This rule, translated in ordinary vocabulary, states that the recording of an event x points to an observation of event x , which actually or supposedly took place and that the observer A produced the recording of the event. Stated simply, every event has its record, and every record implies an event which took place.

Now returning to $A T x = S x + V x$, the internal support function S is preserved, while for the proof $Q x$, is substituted the label $V x$: " x is true". Later on the terminology "mind", M , shall be used for $A T$. Hence,

$$\text{Mind} = M = A T = S + V$$

Combining the two above statements, a most fundamental relationship is derived:

$$\text{Cognitive system} = A = M B = \text{mind operating on body}$$

This follows from: $M B = A T T A = A T A = A$.

One concludes that every cognitive system consists of a part which is called mind and a part which is called body. The mind M contains the support S , which obtains subjective significance to statements x . The body B acts as a receiver and recorder of the procedure Qx which leads to the proof of statement Ax . The mind M also directs the operation Q of the body B to receive or record.

This interpretation brings one close to a full understanding of the internal structure of cognitive systems. The next sections will further elaborate the introduced terminology. First one has to understand how different cognitive systems communicate with each other. This is discussed in the next subsection.

Communication Between Cognitive Systems

Two cognitive systems A_1 and A_2 may communicate. Each system is restricted to making statements of the kind Ax , A_Nx , N_Ax , or T_Ax for an expression of knowledge, which are typical for the quiz situation. Suppose, as before, x is a statement, and speaker A_2 expresses his opinion:

A_2x : "I (A_2) can show x is true"

Now speaker A_1 , upon hearing this, may comment:

$A_1(A_2x)$: "I (A_1) can show: 'I (A_2) can show x is true', is true".

In an earlier subsection, what this statement may signify was discussed. Suppose speaker A_1 has prior knowledge of x , such that A_1x applies, then the above statement $A_1(A_2x)$ is valid, even if A_1 has no knowledge of the method Q_2x by which A_2 intends to prove A_2x (remember $A_2x = S_2x + Q_2x$, where S_2 is the support

and Q_2 is the proof function). Also, since $A_1x = S_1x + Q_1x$, x must belong to the set $\mu(A_1) \cap \mu(A_2)$, i.e., the intersection of sets $\mu(A_1)$ for which A_1x applies, and similarly, $\mu(A_2)$. Hence, x belongs to the set of common knowledge. In this case, no exchange of information has taken place.

The other case is when speaker A_1 wishes to learn by which method Q_2 x , A_2 intends to prove A_2x . Now, A_2 has to make backup statements $A_2A_2 \dashv\dashv A_2x$ which reveal his method and make his knowledge accessible. Now, A_1 , upon receiving this information and finding no fault with it, proclaims $A_1(A_2x)$. For this case, information is passed on from system A_2 to system A_1 .

Some analysis may be helpful here. The algebraic facts of orthogonality between subjective operators S and objective operators Q is needed here.

$$S_1Q_1 = Q_1S_1 = S_2Q_2 = Q_2S_2 = \emptyset, \text{ and } S_1Q_2 = Q_2S_1 = S_2Q_1 = Q_1S_2 = \emptyset.$$

Furthermore, the basic rule of inaccessibility between subjective operators:

$$S_1 S_2 = S_1, S_2 S_1 = S_2$$

and the basic rule of accessibility for objective operators:

$$Q_1 Q_2 = Q_2, Q_2 Q_1 = Q_1$$

apply. Application of these rules gives at once:

$$A_1 A_2 = (S_1 + Q_1) (S_2 + Q_2) = S_1 + Q_2$$

and similarly, for $A_2 A_1$. Notice a curious fact that S_2 and Q_1 have ceased to play a role in the exchange of information.

A little bit of reflection will convince us of the validity of this fact. If A_1 listens to A_2 , A_1 cannot possibly know the subjective feelings or support S_2 , that A_2 will contribute to his knowledge of x . In other words, the subjective support S_2 is inaccessible to A_1 ; it remains within the private domain of A_2 .

On the other hand, if A_2 reveals the procedure Q_2 by which he proves x , that becomes part of the communication process. This explains the basic accessibility of objective processes.

In order to have full access to the procedure Q_2x , A_1 has to be able to understand not only statement x but also the method of proof. This is what S_1x implies; the subjective self S_1 has to be able to interpret the proof Q_2x .

Now, consider the case of a chain of communications: $A_1 A_2 \dots A_n$. From the rules above, one finds: $A_1 A_2 \dots A_n = A_1 A_n = S_1 + Q_n$. This shows the curious fact that all operators except the first subjective support S_1 and the last proof operator Q_n have disappeared.* Only if S_1 can comprehend Q_nx , can the chain of communication said to be successful or to have been validated.

Notice that $A_1 A_2 = A_1 T_{A2}$ which emphasizes the fact that, to the world outside, the system A_2 is presented by T_{A2} : "the truth in A_2 ". Similarly, $A_1 A_{2N} = A_1 N_{A2}$, which shows that "negation in A_2 " is represented by N_{A2} .

Write T_i for T_{Ai} and N_i for N_{Ai} . Then also $A_1 N_2 A_2 = A_1 N_1 A_2 = S_1 - Q_2$, which shows that, in this formula, N_1 and N_2

*This relationship opens up interesting possibility in psychology; i.e., to give an account of Jungian archetypes.

are interchangeable. Also $T_1 T_2 = N_1 N_2 = L_1 + Q_2$. The interpretation of these and similar expressions can become very complicated, since the sets $\mu(L_1)$ (for which $L_1 x$ applies) and $\mu(Q_2)$ (for which $Q_2 x$ is valid) are no longer necessarily separated. Fortunately, the need for such interpretations is minimal.

Next, derive a fundamental relationship which illuminates the process of communication between two cognitive systems. As before, define $M_1 = S_1 + V$ as the "mind" in system $A_1 = M_1 B_1$ and $B_2 = U + Q_2$ as the "body" in system $A_2 = M_2 B_2$; then:

$$A_1 A_2 = M_1 B_2$$

This is, perhaps, the most important result thus far obtained. Applied to a statement x , it signifies that communication is achieved if the mind of system A_1 interprets the statement $B_2 x$, which contains the proof that $A_2 x$ is valid.

Recall that, within the cognitive system A_1 , the mind M_1 also directs and controls the operation of Q_1 , such that $Q_1 x$, the proof that $A_1 x$ applies, becomes available. With communication, information is passed on from A_2 to system A_1 by the mind M_1 simply reading the proof $B_2 x$. In this case, there is no directive procedure on the part of the mind M_1 on operator Q_2 .

Someone might argue that, in order to understand the proof $Q_2 x$, the mind $M_1 = S_1 + V$ may have to exercise a considerable amount of effort. This point is granted and reflects the fact that reading $Q_2 x$ may be difficult; however, what was said is that M_1 cannot control or change $Q_2 x$, since this is supplied by $A_2 x$: "I can show x is true".

Possible applications of the basic formula to hypnosis and quantum mechanics will be discussed in later developments. Also, insight into brain functioning is obtained by using these models. This, in turn, can be used for artificial intelligence modeling.

Growth of Knowledge

More educators will agree that having knowledge is a faculty a student may possess, which could be improved upon. The fact that knowledge can grow or increase points to the need for courses, teachers, and tests to measure the accumulated knowledge. One may expect that greater knowledge provides a better faculty for understanding and for proving contentions.

With this in mind, now embark upon a search for growth factors for knowledge. Compare the two subjective operators S and L, where S provides "understanding" to a cognitive system A and L refers to a "lack of understanding". In between was found the "absolute", general, operator U for "universal being". A natural vantage point is to start with a method that has proven fruitful before:

$$\begin{array}{lcl} \text{Let} & & S = U + s \\ & & L = U - s \end{array}$$

These equations carry much significance. First, consider U and s as new variables which replace the (S, L) pair. Such a transformation is possible, if the properties governing S, L, and U operators remain intact. Secondly, consider the significance of s. The equations strongly suggest that the subjective operator S is obtained from the "neutral" U operator through the addition of a growth component s !

Similarly, the "lack of understanding": L could be interpreted as having a deficiency of growth component s.

Now inquire about the properties s must have. Since $S U = S$ and $U S = U$ define the properties of inaccessibility of subjective operations, this leads to $(U + s) U = U U + s U = U + s$, from which, $s = s U$, $U (U + s) = U U + U s = U$ and $U s = 0$ follow. By using these properties and substitution

into $S S = S$, one finds, similarly: $s^2 = 0$. This last result also follows from the previous two: $s^2 = (s U)^2 = s (U s) U = 0$. Thus, one concludes that s has properties which are quite different from the usual properties which characterize a cognitive system.

A similar argument may be applied to the objective operations. Thus far, only two have been encountered: Q and V . In order to preserve the symmetry between objective and subjective operations, a third operator D is introduced which is defined as follows:

$$\begin{array}{ll} \text{Let} & Q = V + q \\ \text{and} & D = V - q \end{array}$$

Hence, as was done before, the new (V, q) pair replaces the (D, Q) pair by the transformation equations. The operation Q stood for "proof of truth", whereas, V stood for "objective truth", and thus, q can be interpreted as a growth component, which when added to V produces Q . The significance of D (for dumb) now is evident; this operation indicates "lack of proof", or lack of a procedure to produce proof.

Taking this clue, one could define a new cognitive operation: $G = S + D$ (G for "glaube" = "belief" in German), such that Gx : "I do believe that x is true", with the other designation: G_Nx : "I do believe that x is false". Hence, Gx : "I believe x " is to be interpreted as Sx : "I understand x " and Dx : "I have no proof for my belief that x is true". The "believe" concept opens up a whole new class of cognitive systems, where Qx : "I can prove x " is replaced by Dx : "I cannot prove x ". Logically, the systems A and G are quite similar in structure, each having the internal truth and negation operations, etc. The practical significance of the G class does not seem to be as great as with the affirmation class, A , and thus, this case will be left at this junction, to return to the discussion of the former case.

The process of finding rules for the objective growth component q will be continued. Since $Q V = V$ and $V Q = Q$ for objective operations (rules of accessibility), $(V + q) V = V V + q V = V$, from which $q V = 0$, also $V (V + q) = V V + V q = V + q$, and, hence, $V q = q$. Application of these rules and substitution into $Q^2 = Q$ gives an added result: $q^2 = 0$. However, this result is not independent of the previous two, since $q^2 = V (q V) q = 0$. As was the case for the subjective component s , q does not behave as a cognitive operation.

Henceforth, s and q will be called tensions. Notice there are two groups of tensions, the subjective s and the objective q , each with their own characteristic structure. The tensions are the structures which have to be added to the "universal" (U, V) framework in order to produce a cognitive system. This can be seen as follows.

Suppose, in a family of cognitive systems A_i , the universal "truth": $T = U + V$ system applies. Then for A_1 : $A_1 = S_1 + Q_1 = (U + s_1) + (V + q_1) = T + (s_1 + q_1)$. Similarly, we would find for A_2 : $A_2 = T + (s_2 + q_2)$. What A_1 and A_2 have in common is the universal T operation, and since this applies to all A_i of the family, the individual structures which make each cognitive system distinct are the tensions s_i and q_i .

It is a small step from here to consider the (U, V) structure indeed as an already present and existing, truly universal structure! If that were the case, then all that one has to do to build a cognitive system A is to add to this universal structure the tensions s and q .

There is strong evidence that such a universal (U, V) structure indeed exists. Later on, links of cognitive systems with quantum mechanics will support this viewpoint. There are other arguments: If there were no universal structure already present, each individual cognitive system would have to rebuild the (U, V) components on its own. The physical requirements for doing so pose many obstacles, which almost seem to rule out this possibility.

Thus, one is forced to consider the existence of a physical universe in which universal (U, V) components are already present. Later on, what these components are and how they relate to other physical entities will be discussed in detail. Each individual cognitive system is built upon the universal (U, V) structure, and this is what makes communication between cognitive systems possible!

The discussion of growth within a cognitive system will now be completed. We envision a cognitive system $A = (U + s) + (V + q)$, which was built upon a universally existing (U, V), structure by the addition of tensions s and q. The system may be considered "growing" as time passes on, such that, at time t:

$$A(t) = (U + s(t)) + (V + q(t)).$$

What are the properties of tension: s(t) and q(t)? It is possible to expand s(t) and q(t) in a Taylor series: $s(t) = s(t_0) + s^{(1)}(t_0)(t - t_0) + s^{(2)}(t_0) \frac{(t - t_0)^2}{2} + \dots$ where $s^{(1)}(t_0)$ refers to differentiation of s with time, at time t_0 . For short, s(t) and q(t) will be written:

$$\begin{aligned} s(t) &= s_0 + s_0^{(1)} \Delta t + s_0^{(2)} \frac{\Delta t^2}{2} + \dots \text{ and} \\ q(t) &= q_0 + q_0^{(1)} \Delta t + q_0^{(2)} \frac{\Delta t^2}{2} + \dots \end{aligned}$$

Now, s(t) as well as s_0 have to satisfy the two basic properties $s U = s$ and $U s = 0$: and for q(t) and q_0 : $q V = 0$ and $V q = q$ will hold. Substitution of the series into these requirements gives:

$$s_0^{(n)} U = s_0^{(n)}, \quad U s_0^{(n)} = 0$$

for the subjective tensions and

$$q_o^{(n)} v = 0, \quad v q_o^{(n)} = q_o^{(n)}$$

for the objective tensions where $n = 0, 1, 2, 3, \dots$.

Notice that each coefficient in the Taylor-series expansion satisfies the tension rules separately; hence, all time-derivatives of tensions are themselves tensions.

Notice in passing that, if s_1 and s_2 are any subjective tensions, $s_1 s_2 = (s_1 U) s_2 = s_1 (U s_2) = 0$ and, similarly for the objective tensions, $s_1 s_2 = 0$; $q_1 q_2 = 0$. Also, the reader is invited to prove the cross-product rules $s q = q s = 0$ for any s and q type tensions.

These results may be summarized as follows: knowledge exists in the form of a cognitive structure, which is obtained through the addition of subjective (s) and objective (q) tensions to a basic, subjective (U) and objective (V) superstructure. The resulting subjective (S) and objective (Q) structures are mutually independent. Growth of knowledge is accomplished simply by the independent addition (or subtraction) of s - and q -tensions. Each addition or subtraction represents a complete, new state of knowledge. All tensions are independent from each other and from all cognitive operators in opposite structures. Additional rules within each structure are: $s U = s$, $U s = 0$, and $q V = 0$, $V q = q$. Hence, it is the ability to form tensions which is at the core of the cognitive system.

SECTION III

REPRESENTATION OF COGNITIVE OPERATIONS

Matrix Representation of Cognitive Systems

In this section, most of the analytical results of previous sections will be summarized. Thus far, a cognitive system A has been considered as an operation which allows the system to receive input sentences x and produce output statements of the form $y = Ax$.

In order to represent a system with "knowledge", the transformation A has to satisfy the basic rule: $A A = A$. Later, A was split into a subjective component S and an independent objective component Q , such that $A = S + Q$. The last developments showed that S and Q themselves could be split into a general superstructure (U, V) and tensions s and q , such that $S = U + s$, and $Q = V + q$. For the algebra, it was assumed that conventional associative and distributive laws hold for multiplication and addition of operators. The last assumption is the weakest part of the theory, since some properties of the tensions, $s^2 = 0$, $s U = s$, but $U s = 0$, seem far from conventional, and the validity of the use of associative rules, as in $s^2 = (s U) s = s (U s) = 0$, may be questioned. In order to bolster confidence and assure scrutiny, a favorite trick in operator theory is to use matrix representations.

The operator is represented by a matrix which exhibits all the rules and properties the operator must have. The advantage of this procedure is that, since matrices consist of blocks of numbers, every step can be verified through simple calculation with numbers. As shall be seen, other advantages result from the matrix representation. The whole mathematical structure is opened up as it were, and some results not foreseen or not apparent from the operator theory are laid bare.

First, some simple rules for matrices are reviewed. A matrix A consists of a block of numbers, a_{ij} , where $i = 1, 2, \dots, n$ indicate the rows and $j = 1, 2, \dots, n$ labels the columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad (1)$$

Matrix-multiplication: $C = A \times B$ is effected by the rule:

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} \quad (2)$$

Notice that this rule essentially is the basic rule for scalar multiplication of vectors. Let $\bar{a} = (a_1, a_2, \dots, a_n)$, and $\bar{b} = (b_1, b_2, \dots, b_n)$; then the inner, or scalar, product between vectors \bar{a} and \bar{b} is:

$$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \bar{b} \cdot \bar{a} \quad (3)$$

If \bar{a} and \bar{b} are orthogonal, $a \perp b$, then $\bar{a} \cdot \bar{b} = 0$. Hence, every element of matrix C above is the scalar product of a row vector of matrix A and a column vector of matrix B.

The rule for addition of matrices $C = A + B$ is simply effected by:

$$c_{ij} = a_{ij} + b_{ij} \quad (4)$$

The great advantage of calculations with matrices is that they satisfy associative rules, $A(BC) = (AB)C$ for multiplication and conventional distributive laws for numbers apply:

$$A (B + C) = A B + A C \quad (5)$$

Rules exist for finding an inverse A^{-1} (but an inverse does not always exist) such that $A A^{-1} = A^{-1} A = I$ where I is the unit matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & - & - & 0 \\ 0 & 1 & 0 & - & - & 0 \\ - & - & - & - & - & 0 \\ 0 & 0 & 0 & - & - & 1 \end{bmatrix} \quad (6)$$

which has numbers 'one' only in the main diagonal.

One conventional rule not applicable is the commutative rule, which applies to numbers $a b = b a$, but not for matrices. In general: $A B \neq B A$.

Now consider a special type of matrix:

$$D = \bar{a} \times \bar{b} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & --- & a_1 b_n \\ a_2 b_1 & a_2 b_2 & --- & a_2 b_n \\ - & - & - & - \\ a_n b_1 & a_n b_2 & --- & a_n b_n \end{bmatrix} \quad (7)$$

Notice that all elements of matrix D consist of products of two numbers which are taken from the components of vectors \bar{a} and \bar{b} . This special arrangement of numbers is called a dyadic, or outer product of two vectors. Since outer products are matrices, they follow all the above rules for matrices. In addition, it is easy to check the following simplified multiplication rule:

$$(\bar{a} \times \bar{b}) (\bar{c} \times \bar{d}) = (\bar{b} \cdot \bar{c}) (\bar{a} \times \bar{d}) \quad (8)$$

This rule shows that, with outer product multiplication, the two "inner" vectors are used for the scalar product which is a single number, while the two outer vectors are preserved to form the vectors for the new outer product. These properties indicate that outer products are particularly useful to represent the idempotent operators which appear in cognitive systems.

Let us consider $A A = A$, the fundamental rule for a cognitive system A . Let $A = \bar{a} \times \bar{b}$. Then

$$A A = (\bar{a} \times \bar{b}) (\bar{a} \times \bar{b}) = (\bar{a} \cdot \bar{b}) (\bar{a} \times \bar{b}) \quad (9)$$

Hence, $A A = A$ applies only if $(\bar{a} \cdot \bar{b}) = 1$. This is the only constraint on the operator representation $A = \bar{a} \times \bar{b}$.

Recall that, for subjective operators, $S_1 S_2 = S_1$ and, for objective operators, $Q_1 Q_2 = Q_2$. First, let $S_1 = \bar{c}_1 \times \bar{d}_1$ and $S_2 = \bar{c}_2 \times \bar{d}_2$. Then

$$S_1 S_2 = (\bar{c}_1 \times \bar{d}_1) (\bar{c}_2 \times \bar{d}_2) = (\bar{c}_2 \cdot \bar{d}_1) (\bar{c}_1 \times \bar{d}_2). \quad (10)$$

Hence, $S_1 S_2 = S_1$ only if $(\bar{c}_2 \cdot \bar{d}_1) = 1$ and $\bar{d}_1 = \bar{d}_2$. A little reflection will indicate that since S_2 could be replaced by any S_i for which $S_1 S_i = S_1$, the condition $\bar{d}_1 = \bar{d}_2 = \bar{d}_3 = \dots = \bar{d}_i$ would exist for any i . It follows that \bar{d}_i must be a universal vector. Let us call $\bar{d}_i = \bar{u}$. Now, the scalar condition becomes $(\bar{c}_2 \cdot \bar{u}) = 1$, and since this must apply to all \bar{c}_i , we have $(\bar{c}_i \cdot \bar{u}) = 1$.

In order to facilitate the understanding of the scalar product condition, it is useful to consider vectors \bar{c}_i as having one component in the direction of \bar{u} and one orthogonal to \bar{u} . Let $\bar{c}_i = \alpha_i \bar{u} + \bar{s}_i$, where $\bar{s}_i \perp \bar{u}$. Then the condition is simplified to: $\alpha_i (\bar{u} \cdot \bar{u}) = 1$. Hence, α_i must be constant. Take $\alpha_i = 1$, such that $(\bar{u} \cdot \bar{u}) = 1$ and \bar{u} becomes a normalized universal vector: \hat{u} .

Combining all these results, a complete description is found for S_i :

$$S_i = (\hat{u} + \bar{s}_i) \times \hat{u} = (\hat{u} \times \hat{u}) + (\bar{s}_i \times \hat{u}) \quad (11)$$

Now return to the operator calculus to obtain the basic decomposition of S_i into a universal structure U and a tension s_i : $S_i = U + s_i$. A comparison with the outer product equation above gives two identities:

$$U = \hat{u} \times \hat{u} \quad (12)$$

for the universal operator with $(\hat{u} \cdot \hat{u}) = 1$ and

$$s = \bar{s} \times \hat{u} \quad (13)$$

for the subjective tensions, where $\bar{s} \perp \hat{u}$.

An analysis completely analogous to the above, which will be left to the reader, will yield, for the objective operators:

$$Q_i = \hat{v} \times (\hat{v} + \bar{q}_i) = (\hat{v} \times \hat{v}) + (\hat{v} \times \bar{q}_i) \quad (14)$$

Here \hat{v} is a normalized, universal vector, and the \bar{q}_i are vectors orthogonal to \hat{v} : $\bar{q}_i \cdot \hat{v} = 0$.

Earlier, the operators $Q_i = V + q_i$ were discussed, where V was the universal objective operation for 'truth' and q_i were objective tensions. Now V and q_i are identified from the above representation:

$$V = \hat{v} \times \hat{v} \quad (15)$$

for the universal objective operators, with

$$(\hat{v} \cdot \hat{v}) = 1 \quad \text{and}$$

$$q = \hat{v} \times \bar{q} \quad (16)$$

for the objective tension, with $\bar{q} \perp \hat{v}$.

Now it can easily be verified that, indeed, $q \cdot V = 0$ and $V \cdot q = q$ are satisfied:

$$q \cdot V = (\hat{v} \times \bar{q}) \cdot (\hat{v} \times \hat{v}) = (\bar{q} \cdot \hat{v}) (\hat{v} \times \hat{v}) = 0, \quad (17)$$

since q and \hat{v} are orthogonal ($\bar{q} \cdot \hat{v} = 0$). Similarly,

$$V \cdot q = (\hat{v} \times \hat{v}) \cdot (\hat{v} \times \bar{q}) = (\hat{v} \cdot \hat{v}) (\hat{v} \times \bar{q}) = \hat{v} \times \bar{q} = q. \quad (18)$$

The reader is invited to verify that $V^2 = V$ and $q_i \cdot q_j = 0$. The same procedures would show $s \cdot U = s$, $U \cdot s = 0$, and $U^2 = U$ for the subjective components.

Notice that the rules for the possibility of intercommunication, $S_i \cdot S_i = S_i$ and $Q_i \cdot Q_i = Q_i$, between cognitive systems necessarily precondition the existence of a universal superstructure (U, V).

Since objective and subjective systems are independent, it is natural to expect $\hat{u} \perp \hat{v}$. This follows easily from $U \cdot V = 0$. Also, from this, $U \cdot q = 0$ and from $q \cdot U = 0$, follows: $\bar{q} \perp \hat{u}$ as is to be expected. Similarly, $\bar{s} \perp \hat{v}$ would follow from $V \cdot s = 0$, and $\bar{s} \perp \bar{q}$ would follow from $Q \cdot S = 0$.

A remarkable property is discovered: \bar{s}_i , \bar{q}_i , \hat{u} , and \hat{v} are mutually orthogonal 'vector spaces'. Whereas, \hat{u} and \hat{v} are constant universal unit vectors which determine that superstructure, \bar{s}_i and \bar{q}_i , must be interpreted as mutually orthogonal subjective and objective vector-spaces, which is also indicated by S and Q . Each individual subjective and objective "state" which makes up a cognitive state A_i is now represented by particular vectors \bar{s}_i and \bar{q}_i in these spaces. Further discussion of the significance of these vectors is given later.

The above interpretation covers all properties of the cognitive operators and tensions discussed in previous sections. These properties are summarized in Table 6. Notice that the neutral operation $n = v \times u$ is not included in the table. This operator stands for "necessity" and plays a role in modal logic (see Zeman [3] and Prior [4]). One defines operators $L_z = T + n$ (for necessity truth) and $M_z = T - n$ (for possible truth). It is easy to show that n is a neutral tension ($n U = n$, $U n = \phi$, $n V = \phi$, $V n = n$, and $n^2 = \phi$). Furthermore, $n T = n$, $T n = n$, $n N = -n$. From this it is easy to show $M_z = N L_z N$.

Structure of Knowledge and Statements

The previous subsection demonstrated that the outer product representation of a cognitive system A revealed a very simple basic structure consisting of four mutually orthogonal vector spaces. We now carry the discussion a step further to include the statement x to which the cognitive function applies itself. What could be the representation of x ?

Recall that $A = \bar{a}_1 \times \bar{a}_2$ is an outer product representation for which $A A = A$, such that $(\bar{a}_1 \cdot \bar{a}_2) = 1$ must be satisfied. Now a representation for $y = A x$ must be found, where both x and y are statements. A natural and simple solution which fits into the framework of outer product representation is to represent x as a vector: \bar{x} . Then, $y = A x = (\bar{a}_1 \times \bar{a}_2) \bar{x} = (\bar{a}_2 \cdot \bar{x}) \bar{a}_1$, and indeed, vector \bar{x} is transformed into vector $\bar{y} = (\bar{a}_2 \cdot \bar{x}) \bar{a}_1$.

This was the bonus result of the outer product representation: It leads to the unexpected result that a statement can be represented mathematically simply by a vector!* The full significance of this result will now be investigated. Since $A = S + Q = (U + s) + (V + q)$, the outer product representation of Ax is:

*This result should be useful in linguistics research.

$$A x = \{(\hat{u} + \bar{s}) \times \hat{u} + \hat{v} \times (\hat{v} + \bar{q})\} \bar{x} \quad (19)$$

$$= (\bar{x} \cdot \hat{u}) (\hat{u} + \bar{s}) + (\bar{x} \cdot (\hat{v} + \bar{q})) \hat{v} \quad (20)$$

The result shows that, if $y = A x$, the vector representation \bar{y} of y has components in three mutually orthogonal directions: \hat{u} , \hat{v} , and \bar{s} (but not \bar{q}). A host of other properties can be read from the basic result above.

First consider $U x = (\hat{u} \times \hat{u}) \bar{x} = (\bar{x} \cdot \hat{u}) \hat{u}$. This is recognized as the component of \bar{y} in the \hat{u} direction. Before, $U x$ was interpreted as "x exists" \therefore . We have the rather surprising result that "x exists" is measured by $(\bar{x} \cdot \hat{u})$, which is the component of \bar{x} in the \hat{u} direction! Let $\sigma = (\bar{x} \cdot \hat{u})$, then σ represents a signal that x exists; σ is called signal of x. The signal supplies the information that "x is there", or that x exists.

The equation for \bar{y} shows that σ also is the component of \bar{y} in the \bar{s} direction. Now, since $S x$ was identified with "I understand x", $S x = \{(\hat{u} + \bar{s}) \times \hat{u}\} \bar{x} = \sigma (\hat{u} + \bar{s})$, gives the components of \bar{y} in the \hat{u} and \bar{s} directions. Hence, \bar{s} makes up the component of subjective self, or "I" in "I understand x", and $\sigma \bar{s}$ indicates that "I can read the signal of x". This comprises the purely subjective aspect of "knowing x".

The purely objective part is given by the component of \bar{y} in the \hat{v} direction. Apparently, statement x has aspects or components which relate to the \hat{v} and \bar{q} directions! The \hat{v} direction is "universal", whereas, \bar{q} is determined by the system A. The simplest to explain is the part $(\bar{x} \cdot \bar{q})$, since this clearly indicates a choice on the part of system A to accept and interpret parts of statement x . This is seen as follows. Suppose statement \bar{x} , which has a component σ in the \hat{u} -direction and a component δ in the \hat{v} -direction, also has other aspects, features, or parameters which are represented by a set of numbers

p_i . Then $x = (\sigma, \delta, \bar{p})$, where $\bar{p} = (p_1, p_2, \dots, p_n)$ now represent the features of \bar{x} as a vector of \bar{p} in a parameter space P .

Therefore, since \bar{q} is orthogonal to the \hat{u} and \hat{v} directions

$$(\bar{x} \cdot \bar{q}) = (\bar{p} \cdot \bar{q}) = p_1 q_1 + p_2 q_2 + \dots + p_n q_n. \quad (21)$$

This equation may be interpreted as the sensors of system A having assigned weights q_i to the features of x . The weights relate to the significance A has attached to the different features of x .

The scalar product describes the account system A has given of statement x . Now, since Qx : "I can prove x ", and $V x = (\hat{v} \times \hat{v}) \bar{x} = \delta \hat{v}$: " x carries the label true", are identified, $Qx = \hat{v} \times (\hat{v} + \bar{q}) \bar{x} = (\delta + \bar{p} \cdot \bar{q}) \hat{v}$ and the identity holds: "I can prove x = " x carries the label true" and "I can give an account of x ". This explains the three aspects or types of components $(\sigma, \delta, \bar{p})$ of \bar{x} .

It is through the operation of \bar{q} that system A has control over the choice of features of x to show the validity of $Q x$: "I can prove x ". The q_i may also be interpreted as coordinates of a vector \bar{q} in configuration space Q . This interpretation gives A a choice of location \bar{q} in configuration space Q , and $(\bar{p} \cdot \bar{q})$ assigns to this location the interaction with features p_i of x . The coordinate vector \bar{q} also may be viewed as a process, or program, consisting of q_i steps, and $(\bar{p} \cdot \bar{q})$ is the net result of subjecting input x to the program, or the output, which confirms the statement $Q x$: "I can prove x ".

Each particular application will determine whether q_i is considered as a set of sensors with weights, as coordinates, as steps in a program, or perhaps some other interpretation is needed.

This completes the account of the formulas. There still remain unanswered questions. What is the signal σ of a statement? According to the theory, the 'signal' is a universal property of any event or statement. But also, we have seen explicitly that this component of x is given in a direction orthogonal to the objective features p_i . Hence, it must be an elusive property, perhaps not objectively attainable, but accessible only to the subjectivity of a subject. The question is left open at this juncture, until a more general understanding of the processes of knowledge which relates to x as an event or as a reference to an event is obtained.

Another question is concerned with the features p_i of a statement. What are the features of a statement x ? If the statement is a theorem or proposition which requires a procedure for proof, the q_i determine the steps of the procedure and thus p_i are those aspects of statement x which are used to complete the proof. For a mathematical theorem, consider q_i the lemmas applied to parts p_i of the theorem which are used to produce proof. The significance of δ is simply that of designation, label, or title. If x is: "the Pythagorean theorem", then δ is the title: "Pythagorean theorem" by which x : "the Pythagorean theorem" is recognized.

The basic equation for a spoken assertion $y = A x$ indicates that "pure sentences" x and spoken assertions y must be distinguished from one another. A pure, or general, sentence x has, aside from the two components: σ in the \hat{u} direction, and δ in the \hat{v} direction, several, and perhaps a large number of features p_i , whereas, \bar{y} has components only in \hat{u} , \hat{v} , and \bar{s} directions. Also, since \bar{s} is inaccessible to any outside interpreter, only the \hat{u} and \hat{v} components of \bar{y} are effective in communication.

This is exactly what $B x = (U + Q) x$ expresses. For this reason, the expression

$$B x = \sigma \hat{u} + (\delta + \bar{p} \cdot \bar{q}) \hat{v} \quad (22)$$

is called the record of x . The record is simply the spoken assertion of x as it is interpreted by some spokesman. This is underscored by the formula: $A_1 A_2 x = M_1 B_2 x$, where A_1 's "mind" interprets or reads the record $B_2 x$ of A_2 's assertion.

The distinction between "pure" sentences or events and "records" of sentences or events is of fundamental significance in our development of the theory and its applications. The distinction is that a pure sentence or actual event can be acted upon by the cognitive system A , whereas a record of an event can only be read, replayed, watched, etc., and, hence, is passive. In the former case, A acts with \bar{q} on features \bar{p} of \bar{x} to record $(\bar{p} \cdot \bar{q})$. In the latter case, $(\bar{p}_2 \cdot \bar{q}_2)$ has been recorded by A_2 and A_1 can read the record but cannot alter its content.

General Theory of Cognitive Systems

The structure of a finite cognitive system of the affirmation type A has been analyzed, and when it is applied to statement x , a valid and meaningful assertion statements x , such that Ax : "I can show x is true". A large class of processes where knowledge is involved can be covered by this structure, since x could cover questions regarding statements in almost any human field or enterprise.

In fact, this is the basis for the commonly used multiple choice questionnaire, by which knowledge is tested, and on which the formal development of cognitive systems is based. However, the resulting mathematical structure is suspected of covering even a wider field of application.

Perception is one such case, and numerous other activities come to mind: teaching (teacher-student interaction), medicine (brain research), health care, sports, etc., (mind-body interaction), psychology (awareness of self, subjective-objective

interaction), sociology, government and law (expression of value structures in society), science (theory of space-time, quantum theory, observer-event interaction), engineering (pattern recognition, communications, man-machine interaction, artificial intelligence, computer technology, etc.) and even such esoteric fields as development of the mind processes. No doubt the list could be extended almost indefinitely since it could include every biological process or activity where application of knowledge is an essential ingredient. In order to cover the widest choice of applications, x will be called an event. The cognitive system C is said to apply itself to the event.

As usual, the basic rule $C \ C = C$ must be satisfied, and C must have a subjective "support" S and an objective part Q such that $C = S + Q$. The function of the support of "self" S is to bring significance, meaning, and understanding to the system; the "quest", or intelligence Q , interacts with the event. The two component parts satisfy basic rules for inaccessibility: $S_1 S_2 = S_1$ for subjective components and accessibility: $Q_1 Q_2 = Q_2$ for objective components. These basic rules, which make communication between cognitive system C_1 and C_2 possible, also reveal the existence of two independent universal structures U and V . The U -structure establishes actual existence or being, whereas, V -structure acts as a store for objective truth or recorded facts.

Another interpretation for system C : the part

$$M = S + V \quad (23)$$

was called "mind" and

$$B = U + Q \quad (23)$$

was called "body", thus

$$C = M \ B \quad (24)$$

expresses the cognitive system as a "mind" operating on or controlling a "body". A striking rule for intercommunication:

$C_1 C_2 = M_1 B_2$ follows, which states, that if system C_1 "listens" to C_2 , the effect is that of its mind "reading" C_2 's body-sensor operations. Applications to communications, teaching, persuasion techniques, hypnosis, etc., readily suggest themselves. Applications of this rule to quantum mechanics, sociology, management, government and law, as well as TV watching, reading, or hearing are also found.

The mathematical representation theory for system C is completely the same as was developed for system A. One difference to be noticed is that the strict concern for "truth" and "negation" schemes which dictated the development of system A will be less noticeable for the general cognitive system C. The reason for this difference is that, whereas A is closer to logic, the system C is closer to everyday experience, and notions of "truth" are usually taken for granted rather than expressed explicitly.

To illustrate this point, if x is a TV program and Cx : "I am watching x is true", then $C_N x$: "I am watching x is false", $T_C x$: "I am not watching x is false", and $N_C s$: "I am not watching x is true", hardly are everyday concerns or expressions of common experience, except for the first one: $C x$ where usually "is true" is omitted.

The logical structure of truth and negation which determines the structure of C is implicitly understood but not explicitly expressed in most common experiences. For this reason, attention will be devoted to the discussion of $C x$, which is an expression for the experience of event x .

The mathematical representation of C was given in outer product notation as

$$C = (\hat{u} + \bar{s}) \times \hat{u} + \hat{v} \times (\hat{v} + \bar{q}) \quad (25)$$

whereas, event x was given by a vector representation

$$\bar{x} = (\sigma, \delta, \bar{p}) \quad (26)$$

where

$$\sigma = (\bar{x} \cdot \hat{u})$$

and

$$\delta = (\bar{x} \cdot \hat{v}).$$

The result of applying C to x was found as:

$$C x = \sigma (\hat{u} + \bar{s}) + (\delta + \bar{p} \cdot \bar{q}) \hat{v} \quad (27)$$

Recall that the system C is given by the mutually orthogonal vector spaces \hat{u} , \hat{v} , \bar{s} , and \bar{q} . The \hat{u} and \hat{v} spaces are universal and fixed, \hat{u} derives "existence", \hat{v} is called the store where records $(\bar{p} \cdot \bar{q})$ and label δ of the event are kept. The \bar{s} space is variable and is part of the subjective space S .

In other words, for every event \bar{x} , a choice of appropriate \bar{s} may determine its significance for the system C . The \bar{q} space likewise is a variable part of objective coordinate space Q , i.e., each event requires a special selection of \bar{q} by system C to record the event. Similarly, the variable \bar{p} vector is part of the parameter space P which characterizes the occurrences of the event.

The only parameter not mentioned is the "signal" σ which announces the existence of the event. The equation indicates that the signal is accepted by \bar{s} , i.e., the subjective space S will accept x as a meaningful event if an appropriate significance can be attached to σ by a suitable choice of vector \bar{s} . This experience is commonly expressed by the phrase: "trying to make sense" out of an event.

The cognitive process is envisioned to take place as follows: The event is first experienced by the body and the mind, which both receive a signal σ announcing that "something is occurring". The mind, through its supporting function \bar{s} , evaluates and interprets the signal and also directs the body with sensors \bar{q} to receive and record the event. If these actions of body and mind are successfully executed, the event is understood and its parameters and features p_i have become known. This information is recorded and kept in the store \hat{v} where it remains accessible to any other "mind" wishing to recall the event. The store may consist of ordinary storage of recorded information, i.e., books, documents, newspapers, notes, records, tapes, files, memory, etc., or it may be the event itself which contains its own record in the store \hat{v} . These points will be discussed in more detail in future developments.

Matched Events and Stokes Vector Representation

In the previous discussion, a general cognitive system C was said to contain two operations called "body" B and "mind" M, such that $C = M B$. The body operates on a given event

$$x = (\sigma, \delta, \bar{p}) \quad (28)$$

such that

$$B x = \sigma \hat{u} + (\delta + \bar{p} \cdot \bar{q}) \hat{v} \quad (29)$$

describes the process. σ is called "signal" for the event, and $(\sigma + \bar{p} \cdot \bar{q})$ is called the "record" which is stored in the \hat{v} channel (\hat{u} and \hat{v} are normalized orthogonal vectors). The record contains the designation symbol δ which names the event and the steps $(p_1 q_1, p_2 q_2, \dots, p_n q_n)$ which led to the completion of the event taken place.

Consider a book as an example for recorded information. The method for recording, which is symbolized by the \bar{q} -vector, is contained by the physical aspects of the book, with its pages, sentences, organization, chapters, etc. The symbolic contents are the features p_i which are contained by the book. The nature of the "signal" is not yet precisely clear, except that it designates "existence" for the book. In other words, σ has to do with the fact that the book is an item of reality for me, the observer. It guarantees that the book is a real event.

Notice that Bx in Equation (29) has two components in \hat{u} and \hat{v} directions, whereas the transformed vector $y = C x$ has other components. In order to conserve symbols, from now on A will be written instead of C for the general cognitive system. For the general transformation $y = A x$,

$$y = A x = \sigma \hat{u} + (\delta + \bar{p} \cdot \bar{q}) \hat{v} + \sigma \bar{s} \quad (30)$$

or, for short

$$y = (\sigma, \delta + \bar{p} \cdot \bar{q}, \sigma \bar{s}) \quad (31)$$

The last notation clearly brings out the transformation properties of system A . Upon comparing Equations (31) and (28), observe that the signal is preserved intact. To the second term of x is added $\bar{p} \cdot \bar{q}$, and the third part, which indicates features \bar{p} of x , is replaced by $\sigma \bar{s}$. The transformed state y will be called the interpretation of event x given in system A . An ideal interpretation may be considered such that $y = x$. For such an ideal case, $\bar{p} \cdot \bar{q} = 0$ and $\bar{p} = \sigma \bar{s}$. Obviously, such ideal conditions cannot be realized exactly, but it can be an objective to be attained for any state of knowledge. Such an objective produces greater understanding into the processes of knowledge and how optional conditions can be obtained within a finite system.

In fact, when ideally $\bar{p} \cdot \bar{q} = 0$ and $\bar{p} = \sigma \bar{s}$, the system is said to be matched to the event. The terms $\bar{p} \cdot \bar{q}$ and $(\bar{p} - \sigma \bar{s})$ are then the error terms; they obtain a measure for the mismatch of the system to the event. The second requirement $\bar{p} = \sigma \bar{s}$ gives valuable information, even for the non-matched condition: "the subjective space S, of which \bar{s} is a member, is a subspace of the feature space, P, of which \bar{p} is a member". In order to indicate this fact, henceforth, \bar{p}_A will be written \bar{s} , and in order to conserve symmetry in notation, we write \bar{q}_A for \bar{q} .

Basic insights into the processes of cognition have been obtained. Given an event x , with features \bar{p} , the cognitive function obtains an interpretation y_A for the event, which has the same "signal" σ and interpreted features $\sigma \bar{p}_A$. The curious fact of the interpretation lies with the multiplication factor σ , which appears in $\sigma \bar{p}_A$, but which does not appear in \bar{p} . This discrepancy can easily be resolved by first assuming \bar{p}_A to be a unit vector, \hat{p}_A , and then defining σ as a "length" of vector \bar{p} , such that:

$$\bar{p} \cdot \bar{p} = \sigma^2 \quad (32)$$

Associated with Equation (32), a unit vector \hat{p} can now be defined such that:

$$\sigma \hat{p} = \bar{p} \quad (33)$$

These refinements in notation will have important consequences. First, the event described by Equation (28) will be rewritten as:

$$x = \sigma (\hat{u} + \hat{p}) + \delta \hat{v} \quad (34)$$

and for the interpreted event if Equation (30):

$$y_A = A x = \sigma (\hat{u} + (\hat{p} \cdot \hat{q}_A) \hat{v} + \hat{p}_A) + \delta \hat{v}. \quad (35)$$

The objective of the cognitive function is now even more apparent: it must match as closely as possible the direction of vector \hat{p}_A to the direction of feature vector \hat{p} . The "signal" for event x simply becomes the 'length' of the interpreted feature vector \hat{p}_A .

The task of direction finding of feature vector \hat{p} is analogous to obtaining direction information of targets in space by radar. The task of finding target-direction in space becomes equivalent to matching the vector \hat{p}_A to \hat{p} , where \hat{p}_A now indicates the direction of the beam produced by the radar antenna, similar to the familiar searchlighting operation on a dark night in the sky. The analogy with radar direction finding of targets will be pursued further in the following sections.

The designation δ in Equations (34) and (35) can be any arbitrary code by which the event is named and can be recalled. It serves no further purpose and can be put aside for most theoretical developments. Hence, it is convenient to write for Equation (34):

$$x = p + \delta \hat{v} \quad (36)$$

where

$$p = \sigma (\hat{u} + \hat{p}) \quad (37)$$

Frequently p will be used instead of x to denote the event. The vector p has important properties which are basic to the understanding of cognitive systems and its connection with quantum theory (wave mechanics). If written

$$p = (p_u, \bar{p}), \quad (38)$$

where

$$p_u = \sigma \text{ and } \bar{p} = \sigma \hat{p}, \text{ then:}$$

$$p_u^2 = \bar{p} \cdot \bar{p} . \quad (39)$$

This follows easily from the definitions above and Equation (32). The innocent looking relationship of Equation (39) carries a lot of weight, as we shall see later.

The reason why "length" was parenthesized with Equation (32) is that Equation (32) also will be used if vector \bar{p} is complex, which is indeed unusual, since normally one would expect $\sigma^2 = \bar{p} \cdot \bar{p}^*$ to be true, where $*$ denotes complex conjugation. The property expressed by Equation (39) also for complex \bar{p} is peculiar to a mathematical object called a Stokes vector.^{*} A basic result is that, if a designation is omitted, all events are described by Stokes vectors. In addition to this result, system A also contains vectors $p_A = \hat{u} + \hat{p}_A$ and $q_A = \hat{v} + \bar{q}_A$ such that now:

$$\Lambda = (p_A \times \hat{u}) + (\hat{v} \times q_A) \quad (40)$$

and

$$y = Ap = \sigma p_A + \sigma (\hat{p} \cdot \bar{q}_A) \hat{v} \quad (41)$$

For the matched condition, $p \cdot \bar{q}_A$ can be expected to be small and $\sigma = 1$, such that $Ap = p_A$, which shows one Stokes vector p being "interpreted" by another: p_A .

Since

$$Bp = \sigma (\hat{u} + (\hat{p} \cdot \bar{q}_A) \hat{v}) \quad (42)$$

and $p \cdot \bar{q}_A$ is small, Equation (42) can be rewritten approximately as an exponential

$$Bp = \sigma e^{i \hat{p} \cdot \bar{q}_A} \quad (43)$$

^{*}Reference 8

where (\hat{u}, \hat{v}) for this occasion is replaced by the complex-plane $(1, i)$ coordinates.

The result of Equation (43) is highly significant: it gives an identification of the "super-structure" vectors \hat{u} and \hat{v} as real and imaginary components in a complex plane representation of an exponential function, and secondly, Equation (43) suggests a description of the body function for the more general case, when $\hat{p} \cdot \bar{q}_A$ is not necessarily small, i.e., the general target search procedure. All these concepts will be dealt with in greater detail in later subsections.

Summary of Status and New Developments

In the last subsection, an important juncture of our development of cognitive systems was presented. A cognitive system A , for which $A^2 = A$ holds, describes a system in which a match between "body" and "mind" functions is achieved which remains intact as a condition for cognitive functioning. The condition is that the vectors \hat{p}_A and \bar{q}_A of mind and body functions at all times are orthogonal: $\hat{p}_A \cdot \bar{q}_A = 0$.

Also, when confronted with an object-event described by features \hat{p} , $\bar{q}_A \cdot \hat{p}$ is not generally zero, but an attempt is made to achieve the ideal matched condition, when $\hat{p} \cdot \bar{q}_A = 0$. Thus, the distinction between the working of the internal system and the external search function is made clear by these simple relationships.

From now on, the convention will be adopted that the statement $y = Ax$ only makes sense if $\hat{p} \cdot \bar{q}_A$ is small, i.e., in the neighborhood of perfect match. In that case, Bp can be defined rigorously as

$$Bp = e^{i \hat{p} \cdot \bar{q}_A} (\hat{u} + (\hat{p} \cdot \bar{q}_A) \hat{v}) \quad (44)$$

which is the (complex) received signal from the target-event.

This case is then easily generalizable to describe the complete search function when Equation (44) on the left becomes the description of the received signal during the entire search mode. It should be kept in mind, however, that in making this step, the strict adherence to the cognitive structure $A \rightarrow A$, etc., must be abandoned since this structure becomes applicable only when the search for the target-event has resulted in a satisfactory match, i.e., the target has been found.

The cognitive function thus applies to the target tracking mode of radar, after the search has resulted in target acquisition. Only then does knowledge exist about the position of the target. (Position is here defined by vector \hat{p} at time t .)

The junction we were speaking about consists of proceeding with the new task of analyzing the body-function $B(p) = \sigma e^{i \hat{p} \cdot \vec{q}} A$ during the entire search procedure which leads to target acquisition. The analogy with radar will be pursued in greater detail in the following subsections.

The previous junction also leads to a connection with quantum mechanics. This road connects by means of the result obtained in the last subsection that the observed event is given in the form of a Stokes vector. The Stokes vector automatically defines a Hamiltonian equation (this follows from: $p_u^2 = \vec{p} \cdot \vec{p}$). The Hamiltonian equation and resulting form for the Hamiltonian, which determines classical Newtonian and relativistic mechanics, is then shown to lead to the well-known theory of elementary particles.

From this point of view, the process $B(p) = \sigma e^{i \hat{p} \cdot \vec{q}} A$ gives the interaction of an object with features (momenta) p with a "measuring apparatus", given by the "body": B . The result is that $B(p)$ describes the "reduction to a wave packet" of the object, which expresses the fact that a set of compatible eigenvalues of the object's parameters p has been observed. In order to make the last statement comprehensible to a reader not

familiar with quantum theory, an introduction and exposition of these topics will be developed. This will comprise the avenue of the second branch of our juncture.

The fact that there is a connection between cognition theory and quantum theory, or wave mechanics, is clear from several points of view. First, there is the symmetry which exists between p - and q -variables, which occurs not only in quantum theory but already in classical mechanics and is called Hamiltonian theory. Secondly, the form of $B p$ strongly suggests a wave function if one of the coordinates in q_A is the time-variable ($p \cdot q_A = \omega t - kx$ is the familiar exponent of a progressive wave front).

The simple fact that the wave theory introduces time and space, as part of the observer's q -frame, in turn, has interesting and important consequences for the theory of cognition. This idea can again be turned around to reflect on problems of quantum measurement theory. One of the basic problems still unclear is: how does the object obtain its set of coordinates?

Somehow, one feels that the investigating subject with measuring apparatus is at least partly responsible for implanting a coordinate scheme on the object-event.* That this must be so is clearly indicated by the analogy through symmetry with the "mind" function. Just as the object's features p induce an interpretation vector p_A to the mind, equivalently the body's coordinates q_A introduce coordinate space q for the event! The fact that p and p_A , q and q_A are only approximately the same introduces many interesting consequences for measurement theory and the resulting coordinate transformation laws.

*See Reference (9), p. 291.

SECTION IV

COGNITION AND MATCHED FILTER RECEPTION

Radar Target Detection Analogue

The preceding subsections hinted at an analogue which must exist between perception and detection of targets with radar. That this must be so is not altogether surprising because the radar target detection problem may well be viewed as a special case of perception.

The "body-sensor" in this case is the radar instrument with its antennas for transmission and reception of electromagnetic radiation. The target, of course, is an object in space which the radar is designated to locate. First, it must be established which elements in the analogy are generalizable to perception and which are peculiar to the radar system itself.

The target usually is at a significant distance from the radar site location, such that the antenna system may be considered small compared to the target range, R . Also, it is necessary to distinguish between so-called passive targets which do not, by themselves, produce radiation and active targets such as a beacon which do. For the case of passive target reception, the radar antenna, considered as a point source of electromagnetic (EM) radiation, transmits a spherical wave which has a directive beam pattern $\phi(p)$. In the far field, where the target is located, the spherical wave can be considered as a plane EM wave which strikes the target and produces a return that is captured by the radar-receiver antenna. The target return signal is recorded and processed for the purpose of target detection and location.

For active targets (beacons), only the receiver of the radar is operative. In most cases of passive target reception, the target return is mixed with components of active external noise sources, and a so-called matched filter is employed in the receiver system in order to optimize the target return signal. The received signal which is produced at the matched filter

receiver output can be shown to be equal to the autocorrelation function for the transmitted waveform.* The target signal is then said to be optimally received when the autocorrelation function reaches a maximum. These concepts are very important for generalization to perception. The autocorrelation will be shown to be equivalent to the "body function" B_p of perception.

The physical significance that can be attached to the parameters involved in the comparisons must be kept in mind in order to understand the mathematical analogies. For example, in radar, an electromagnetic wave serves as a connecting link between target and observer. What could be the corresponding link operation in perception? All kinds of physical wave phenomena which could serve this function may be imagined. But very likely many practical obstacles will be encountered because such physical sources may not have been observed experimentally.

Fortunately, the example of quantum mechanics can lead out of this dilemma. There the wave function does not express a physical entity such as an acoustic, hydrodynamic, or electromagnetic wave, but rather a conceptual wave! The conceptual wave structure $\phi(p)$ is interpreted such that the absolute value $|\phi(p)|^2$ gives the probability (density) that the particle, if measured, will be found at location p . In the usual interpretation, p represents a point in the particle's momentum space.

Momentum space p , rather than the particle's ordinary physical space q , was chosen because of developments in perception. In perception, a target is visualized with parameters $\bar{p} = (p_1, p_2, \dots, p_n)$ being observed by a conjugate set of body-sensors $\bar{q}_A = (q_1, q_2, \dots, q_n)$. In radar, the corresponding desired parameters are the target's spatial position variables: $\bar{p} = (p_1, p_2, p_3)$. Some discretion is necessary because in radar, as in cognition, only target

*This is the general case for Gaussian noise sources. See Reference 10, pp. 3-32.

direction is observed in space \hat{p} . Target range is produced in radar by a separate measurement of the time it takes for the signal to make the return trip: $\tau = 2R/c$, where c is the velocity of light. However, no time-delay measurement is assumed to occur with perception. Hence, the "length of p " does not measure target distance, but rather: $\sigma = \text{"strength of } p" = |\bar{p}|$.

The "signal" is called σ , belonging to the target, and σ relates to target existence rather than to the target's spatial distance. σ may be interpreted as a measure of how well the radar succeeds in locating the target's direction in space.

The way in which a radar system operates to locate a target may be explained in several steps. Several modes of operation are used to narrow down the target's coordinates. The first step is to listen for a signal which would indicate that there is a target. The search mode is used to establish target existence. It employs a broad beam, such that a large volume of the physical space is covered. Any target located within this volume will produce a return signal which reveals its presence.

The next step consists of a scanning operation with a narrow beam antenna which is used to narrow down the target's position coordinates. Once the approximate target location is found, the target tracking mode of operation is used to ensure target acquisition within a designated target space or bin. If necessary, a scanning operation with an even sharper antenna beam results in narrowing down the target position into a smaller bin. Every search operation hence results in more precise target positioning.

Note that the above procedure is quite analogous to the processes used in perception to locate an object. First, we "listen" for any sign of existence for the object. An "open-minded" attention mode of operation is used for this listening phase. After something is heard, then the possibilities are narrowed down by concentrating attention focus on specific observables relating to the object. After this, the object is identified. The identification consists simply of correlating

the measured target features with known features of related objects. In the radar problem above, the features measured are the coordinates which determine target direction only, while σ relates how well the beam is pointed at the target, i.e., if $\sigma = 1$ (maximum), the target's position has been found.

The demand for precise target location is dictated by the requirements of the problem to be solved. In a hostile encounter, an enemy target is to be intercepted, and this objective can be met by providing an accurate target location. The same principles apply in perception. The more essential it is for a life situation to know the precise extent of an event which is taking place, the more the mind will concentrate to produce an accurate account of the event.

In the next subsection, a more detailed analysis of these processes will be given.

Analysis of Radar Reception Principle

In this subsection, the radar target detection analogue will be developed in more detail. The analogy is very important for perception and for cognition, in general, because it presents a prime model or example from which all other cases can be derived. Radar operates with electromagnetic radiation as the communication link between the target and the observer. This has to be reinterpreted for perception, and here the quantum model is useful because it introduces instead a conceptual wave, as was indicated earlier. This model can also be used as a source for interpretation in quantum mechanics.

Essentially, the goal is to describe the processes which lead to knowledge. These processes all aim at object acquisition and identification. Target acquisition for radar will be shown to be similar to object acquisition in perception. Both attempt to "capture" a target by processes of optimization of a received signal. The signal is thought to emerge at the output of an optimum filter receiver which is matched to the target return. This signal takes the form of the autocorrelation function for the target return.

The method by which the radar antenna produces its beam $\phi(p)$ is shown in Figure 2. A two-dimensional image in the plane of the paper is shown here for convenience. The antenna usually is a parabolic reflector, which is not shown in the figure. The flat surface covering the face of the reflector is called the antenna aperture. In the figure, this aperture is shown as a tilted plane, indicated by spatial variables \bar{q} . The tilt angle θ_A determines the beam direction vector \hat{p}_A . The radar beam pattern $\phi(\hat{p})$ is formed by an illumination of the antenna aperture with electromagnetic (EM) radiation. The EM energy emerges from the antenna feed system which is located at the parabolic focal area (not shown in the figure). The distribution of EM field over the aperture plane is given by the so-called primary illumination function $\psi(\bar{q})$.

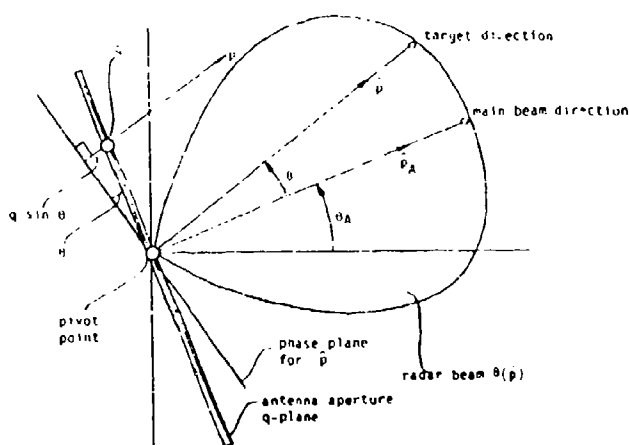


Figure 2. Principle of Radar Beam Formation

In the usual mode of operation, the aperture plane is a surface of constant phase, while the field's amplitude has a smooth, bell-shaped distribution with the maximum field strength at the aperture center. This center is shown here as a pivot point governing the direction θ_A of maximum radiation \hat{p}_A . In actual operation, the mechanical center of rotation may be displaced somewhere on the antenna. This does not alter the

principle of antenna operation involved, since the location of the target is supposed to be in the far field of the beam, i.e., at a distance large compared to the radar antenna dimensions. A consequence of this is that the target direction relative to the radar is given by the unit vector \hat{p} , which aims in the same direction for all points on the antenna aperture.

The beam is produced by adding the contributions of illumination of each point on the antenna aperture to the field in a given far-field target direction. Since radiation is a wave phenomenon, these contributions are added in amplitude and phase. The amplitude* is simply proportional to the illumination pattern $\psi(\bar{q})$. The phase contribution depends on the direction of the target. In the figure, a so-called phase plane is shown orthogonal to the target direction \hat{p} . The phase plane establishes a phase reference for each point \bar{q} on the aperture for the given target direction.

The phase lead for point \bar{q} relative to target direction \hat{p} is shown in the figure as the distance $q \sin \theta$. Notice that for the main beam direction \hat{p}_A all phases of points \bar{q} are zero relative to the antenna aperture plane.

The contribution in amplitude and phase to the far-field at \hat{p} of a point \bar{q} on the aperture thus becomes proportional to

$$F(\theta) = \psi(\bar{q}) e^{ik\hat{p} \cdot \bar{q}} \quad (45)$$

where $k = 2\pi/\lambda$ is the wave number of the phase associated with the field and λ is the wavelength of the electromagnetic radiation. Notice the important fact that the wavelength λ acts as a natural unit of electrical length on the \bar{q} aperture. Since frequency $\nu = c/\lambda$, where c is the velocity of light, a higher

*Amplitude is used here in the quantum mechanical sense; it also includes possible phase contributions of the primary illumination function.

frequency produces a smaller unit of length λ and hence an electrically larger radar aperture.

The total field contributed in a direction θ is the sum of all contributions of points on the aperture

$$\phi(\hat{p}) = \int_R \psi(\vec{q}) e^{2\pi i \hat{p} \cdot \vec{q} / \lambda} d\vec{q} \quad (46)$$

where R is the range of integration over the aperture. From the figure, we find for the phase term:

$$\hat{p} \cdot \vec{q} = q \sin \theta. \quad (47)$$

Equation (46) shows that the antenna pattern and illumination function are Fourier transforms of each other. The theory would require the integration in Equation (46) to range over the total coordinate space Q ; this can be accommodated by defining $\psi(\vec{q}) = 0$ to be outside the range R .

As an illustration, take a one-dimensional case of a linear aperture of size $2a$ which has a uniform illumination as indicated in Figure 3(a). The radiation pattern is found from Equation (46):

$$\phi(\hat{p}) = A \int_{-a}^{+a} e^{i k q \sin \theta} dq \quad (48)$$

The integral is found simply from:

$$\begin{aligned} \int_{-a}^{+a} e^{icx} dx &= \left. \frac{e^{icx}}{i/c} \right|_{-a}^{+a} \\ &= \frac{2}{c} \left[\frac{e^{ica} - e^{-ica}}{2i} \right] = 2a \left[\frac{\sin ca}{ca} \right] \end{aligned} \quad (49)$$

Thus:

$$\phi(\theta) = 2Aa \frac{\sin(k a \sin \theta)}{k a \sin \theta} \quad (50)$$

The function defined by Equation (50) is illustrated in Figure 3(b). The function is proportional to the illuminated aperture size and it has a maximum at $\theta = 0$, as was to be expected.

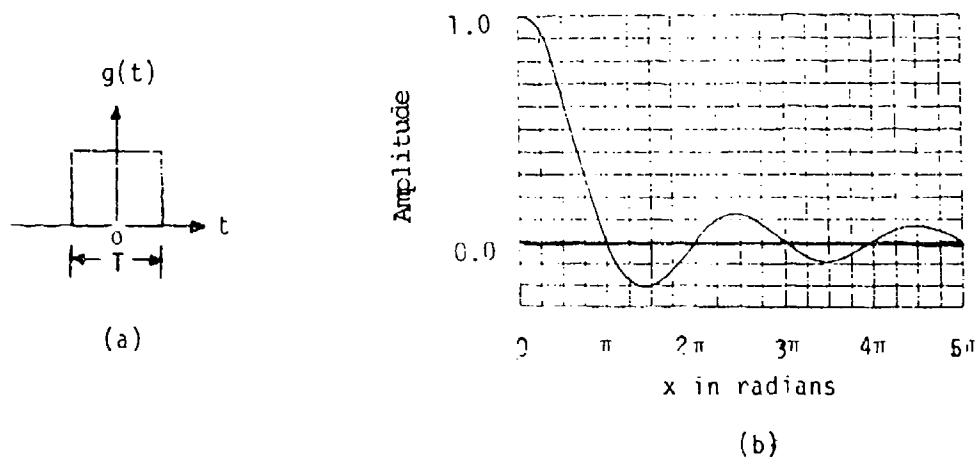


Figure 3. An Illustration of (a) a Uniform Illumination of an Aperture and (b) the Corresponding Radiation Pattern

Notice that in addition to the main beam, several sidelobes with smaller maximum intensity appear. These sidelobes are a general characteristic of antenna patterns and have to be kept small; otherwise, targets appearing in a side lobe direction could cause ambiguous signals which could trigger false alarms.

The width of the main lobe is determined by the electrical antenna size: ka . The direction of the target is given by the angle θ . Notice that the signal produced by the antenna pattern increases as the target direction approaches the direction of \hat{p}_A , where $\theta = 0$. For the condition where $\theta = 0$, the pattern is matched to the target, which means simply that the main beam is pointing in the direction of the target. The sharper the beam shape, the more accurate the target direction is pinpointed by the antenna beam.

If the direction of the target is unknown, a broad beam is used to establish target existence. Once a signal is received, a narrow beam antenna is employed to reduce the uncertainty regarding the location of the target, through pivoting the antenna aperture plane. Several such search procedures may be necessary to pinpoint the target position within the desired accuracy range.

For the two-dimensional case, a rectangular aperture of size $(2a \times 2b)$ is considered with uniform illumination. A slightly more complicated analysis results in the following beam pattern:

$$\phi(\theta, \phi) = 4Aab \frac{\sin(ka \sin \theta \cos \phi)}{ka \sin \theta \cos \phi} \cdot \frac{\sin(kb \sin \theta \sin \phi)}{kb \sin \theta \sin \phi} \quad (51)$$

where ϕ is the angular direction of rotation about the main beam direction \hat{p}_A .

The two so-called principal plane cuts of the beam are obtained for $\phi = 0^\circ$ and $\phi = 90^\circ$. If $\phi = 0$, Equation (51) reduces, except for the constant $2b$, to Equation (50); if $\phi = 90^\circ$, a similar form is obtained with a and b interchanged.

Again notice that the function defined by Equation (51) is proportional to the illuminated antenna aperture area. The width of the lobes in the principal directions are determined by the aperture dimensions $2a$ and $2b$. Taking $(ka \sin \theta_0) = \pi/2$ as a measure of angular width in the principal plane, defined by $\phi = 0^\circ$, an equation for beamwidth follows:

$$\theta_0 = \sin^{-1} (\lambda/4 a) \quad (52)$$

and similarly for the principal plane defined by $\phi = 90^\circ$. If the aperture length $2a$ is large, a small beam angle θ_0 results; if $2a$ is small, a large beam angle is produced.

The same case could be analyzed for a Gaussian beam type of antenna illumination. For the one-dimensional case

$$\psi(q_A) = A/\sqrt{2} e^{-\pi q^2/2 a^2}. \quad (53)$$

Substitution into Equation (46) gives:

$$\begin{aligned} \phi(\theta) &= A/\sqrt{2} \int_{-\infty}^{+\infty} e^{-\pi q^2/2 a^2} e^{i k q \sin \theta} dq \\ &= 2Aa e^{-\frac{(ka \sin \theta)^2}{2\pi}} \end{aligned} \quad (54)$$

where the normalization in Equation (53) is chosen such that Equation (54) closely approximates Equation (50). The total incident power is computed for the two types of illumination in order to compare the two cases.

For uniform illumination, $\psi(q) = A$; hence, the incident power density is $|\psi(q)|^2 = A^2$ and the total power incident on the aperture is

$$\int_{-a}^{+a} |\psi(q)|^2 dq = 2 a A^2 \quad (55)$$

Similarly, for the Gaussian type of illumination,

$$|\psi(q)|^2 = 2A^2 e^{-\pi q^2/a^2} \quad (56)$$

and the total power incident on the aperture is

$$2A^2 \int_{-\infty}^{+\infty} e^{-\pi q^2/a^2} dq = 2 aA^2 \quad (57)$$

The Gaussian type of illumination results in a Gaussian beam-shape as given by Equation (54), which differs from the uniform illumination case by the fact that no minor lobes are present. This accounts for the fact that Equation (54) is only approximately equal to Equation (50). The uniform illumination case is closer to radar engineering design, whereas the Gaussian beam is useful for statistical interpretations.

This completes the introduction to the radar target detection procedure. In perception, one will operate mostly in the active radar reception mode. With sensors q_i as body-receiver system, acting upon target observables p_i resulting in a sequence of measurements: $p_1q_1, p_2q_2, \dots, p_nq_n$, which are combined into the phase term $\bar{p} \cdot \bar{q} = p_1q_1 + p_2q_2 + \dots, p_nq_n$. This would correspond to the measurement of azimuth and elevation angles for the radar target. The secondary field $\phi(p)$ corresponds to the 'attention' field of the observer. Here, the analogy with quantum mechanics is used, where $|\phi(p)|^2 dp$ gives the probability, or estimate, that the target is located within the parameter interval (p and $p + dp$). Because $|\phi(p)|^2 dp$ is now a probability, the integration corresponding to Equation (55) will be normalized:

$$\int_R |\psi(q)|^2 dq = \int_P |\phi(\hat{p})|^2 dp = 1 \quad (58)$$

The target is considered located as a point \hat{p} in $n+1$ dimensional parameter space P . The target return signal is simply the function $\phi(\hat{p})$ of our attention field.

As was mentioned earlier, if external Gaussian noise sources are present, a matched-filter receiver provides optimal reception for this target-event encounter. The so-called received signal at the output of the matched filter will have the form of an autocorrelation function:

$$\begin{aligned}
g(\hat{p}) &= \int |\psi(\bar{q})|^2 e^{2\pi i \hat{p} \cdot \bar{q} / \lambda} dq \\
&= \int \phi(p_1) \phi^*(p_1 + \hat{p}) dp_1
\end{aligned}
\tag{59}$$

This function reaches a maximum for the matched condition, when the direction of maximum attention field \hat{p}_A coincides with the target direction \hat{p} . This state is called the matched condition, i.e., the target has been found and identified.

For the matched condition, the phase term $\hat{p} \cdot \bar{q}$ is small (ideally zero) and, hence, Equation (59) can be written to first order

$$\begin{aligned}
g(\hat{p}) &= \int |\psi(\bar{q})|^2 (1 + 2\pi i \hat{p} \cdot \bar{q} / \lambda) dq = \\
&= 1 + 2\pi i \hat{p} \cdot \bar{q}_A / \lambda = e^{2\pi i \hat{p} \cdot \bar{q}_A / \lambda}
\end{aligned}
\tag{60}$$

where \bar{q}_A is the average q-coordinate of the bell-shaped distribution and is usually located at the center of the radar antenna aperture.

Now, Equation (60) is analogous to the body-function B_p derived in a previous subsection:

$$B_p = \sigma e^{i \hat{p} \cdot \bar{q}_A} = \sigma (\hat{u} + (\hat{p} \cdot \bar{q}_A) \hat{v}) \tag{61}$$

Here, σ corresponds to the "signal" in cognition. From this follows that the cognitive state corresponds to the matched state of target acquisition!

Some important consequences follow for the matched case. For the matched case, the signal is maximum just as it should be, based on the autocorrelation function property. Reference to Equation (60) shows that the maximum has been normalized to unity: $\sigma = 1$. Also observe the important identifications

of \hat{u} and \hat{v} as the real and imaginary axes of an exponential function: $g(\hat{p})$. Because these axes are the same coordinates for the $\phi(p)$ and $\psi(q)$ functions, an important physical identification is obtained: The \hat{u} and \hat{v} axes are the coordinates of the observer's attention field! This identification will have significant implications in quantum mechanics, where the $\psi(q)$ is a well-established physical concept related to observations.

Some simplifying operations are necessary. The imaginary term $2\pi i/\lambda$ in Equation (60) corresponds to \hat{v} in Equation (61). In quantum theory, however, the "length" of pointer vector \bar{p} is usually defined as $|\bar{p}| = h k$, where $h = h/2\pi$ and where h is Planck's constant, instead of $|\bar{p}| = k = 2\pi/\lambda$ as was done in Equation (60). This amounts to a different set of units for the measurement of $|\bar{p}|$.

These units contain in both cases a "wavelength" λ or a frequency $\nu = c/\lambda$, where c is the velocity of light. Hence, the attention field $\phi(p)$ can be considered to propagate with the conventional term $\exp(\omega t - kx)$, where $\omega = 2\pi\nu$. Be aware, however, that "propagation" in this sense occurs in a conceptual space determined by the p and q variables, which makes the idea physically less attractive. This problem also occurs in quantum mechanics if many interacting particles are observed.* The connections with quantum theory are further pursued in the following subsection.

* See Reference 11, p.100.

SECTION V COGNITIVE PROCESSES

Notational Developments Based Upon The Radar Target Analogue Model

The previous subsection has paved the way towards a more complete development of cognitive processes. A close analogy was found to exist between cognitive processes and beam formation principles used for radar detection of objects. For cognition, these beams are represented by so-called attention-fields $\psi(q)$ and $\phi(p)$. The "focus of attention" $\phi(p)$ is directed towards the target, just as occurs with radar observation.

The attention field for the target reception mode is $\phi(p)$, whereas $\psi(q)$ relates to the activation of body-sensors. Between the two attention fields there exists a Fourier-transform relationship:

$$\phi(p) = \int \psi(q) e^{2\pi i p \cdot q} dq \quad (62)$$

and

$$\psi(q) = \int \phi(p) e^{-2\pi i p \cdot q} dp \quad (63)$$

For this case, the units of p and q in the exponential are chosen such that the most convenient representations are obtained.

Notice that Equations (62) and (63) are generally applicable operations which may be used for knowledge formation. Target acquisition is achieved through observation of a matched filter output

$$\begin{aligned} g(p) &= \int |\psi(q)|^2 e^{2\pi i p \cdot q} dq \\ &= \int \phi(p_1) \phi^*(p_1 + p) dp_1 \end{aligned} \quad (64)$$

which is the autocorrelation function for the signal $\phi(p)$ received from the target. This function reaches a maximum when the direction of maximum attention field \hat{p}_A coincides with the target direction \hat{p} .

The "attention fields" are to be thought of as target estimation, or probability density fields, quite analogous to the quantum mechanical state functions. Hence

$$\int |\psi(q)|^2 dq = \int |\phi(p)|^2 dp = 1 \quad (65)$$

In statistics, $g(p)$ is called the characteristic function belonging to the probability density $w(q) = |\psi(q)|^2$. Close to the matched condition, the phase $p \cdot q$ in the integrand of Equation (64) is small, and the following can be written:

$$\begin{aligned} g(p) \quad \mathbb{E} > \int |\psi(q)|^2 (1 + 2\pi i p \cdot q) dq = \\ = 1 + 2\pi i p \cdot q_A \approx e^{2\pi i p \cdot q_A} \end{aligned} \quad (66)$$

where $\mathbb{E} >$ stands for "the process which leads to the cognitive state condition," i.e., to the matched condition; whereas q_A stands for the average q of the distribution. Equation (66) may be looked upon as the basic formula for the "measurement of parameters p ." This will be more evident as the development progresses. Equation (66) describes the basic process of detection, which is the first step towards full target recognition. For the matched condition, it was found that the maximum p_A of the attention function $\phi(p)$ is directed towards the target, i.e., the target has been found and its descriptive parameters have been measured. The well-matching condition now requires that $p \cdot q_A = 0$, which solves for the condition $p = p_A$, since $p_A \cdot q_A = 0$ for the cognitive antenna-sensor system. These topics will be a recurrent theme in subsequent discussions.

The formal connection with cognitive theory, based upon $A \cdot A = A$, is found from the correspondence of Equation (66) with the "body-operation" $B \cdot p$:

$$B \cdot p = \sigma e^{2\pi i p \cdot q_A} = \sigma (\hat{u} + 2\pi \hat{v} (p \cdot q_A)) \quad (67)$$

In Equation (67) the units for p and q were adjusted to correspond with the $2\pi i$ in the exponent of Equation (66), and the requirement $|\bar{p}| = 1$ was dropped momentarily, to be picked up again at a later time.

Hence, as before, one finds the important identification of \hat{u} and \hat{v} as the real and imaginary axes of the complex attention fields. For the matched condition, $\sigma = 1$ as can be seen from comparison of Equations (66) and (67).

The general formalism for attention fields and received signal $g(p)$ will now be written in a form familiar in quantum theory. First, the Dirac notation for scalar product is introduced, i.e.,

$$\langle a | b \rangle = a^* \cdot b = \vec{a}^* \cdot \vec{b} \quad (68)$$

where the star is complex conjugation and the arrows indicate row (bra-) and column (ket-) vectors.

Similarly, the outer product

$$| a \rangle \langle b | = [a_i \ b_j^*] = \vec{a} \cdot \vec{b}^* \quad (69)$$

is defined as the matrix with vector components products as elements. The two notations in Equations (68) and (69) are compatible. For example,

$$(| a \rangle \langle b |)^2 = | a \rangle \langle b | a \rangle \langle b | = (a \cdot b^*) | a \rangle \langle b | \quad (70)$$

which shows the familiar idempotent rule emerging, if proper normalizations are observed.

Following Dirac,* the representation of a ket-vector $|\psi\rangle$ as a set of numbers $\langle \xi' | \psi \rangle$ is introduced where $\langle \xi' |$ is the orthonormal set of eigenvectors which span a space \mathcal{Z} . Belonging to each eigenvector is an eigenvalue ξ' related to the eigenvalue problem

$$\xi |\xi'\rangle = \xi' |\xi'\rangle \quad (71)$$

where ξ is the operator, called an "observable". For the complete set of eigenvectors,

$$\langle \xi' | \xi'' \rangle = \delta(\xi' - \xi'') \quad (72)$$

The above notation can be generalized to the case of a "complete set of commuting observables" $\xi_1, \xi_2, \dots, \xi_n$. If there exists a representation where all observables ξ_i can be made diagonal, then the ket $|\psi\rangle$ will have a representative $\langle \xi_1', \xi_2', \dots, \xi_n' | \psi \rangle$, or $\langle \xi' | \psi \rangle$ for brevity. These representatives may be considered as values taken on by a function $\psi(\xi')$. Hence, we may define

$$\psi(\xi') = \langle \psi | \xi' \rangle \quad (73)$$

Next, apply this notation to the eigenspace of q -variables. First, consider one q -variable (observable) with associated eigenvectors $|q'\rangle$. Similarly, as in Equation (73), a function $\psi(q')$ is defined as

$$\psi(q') = \langle \psi | q' \rangle \quad (74)$$

Now, this formula is substituted into the basic normalization Equation (65)

*See Reference (12), Vol. I, p. 324.

$$\begin{aligned} \int |\psi(q')|^2 dq' &= \int \langle \psi | q' \rangle \langle q' | \psi \rangle dq' = \\ &= \langle \psi | \int | q' \rangle \langle q' | dq' | \psi \rangle = \langle \psi | \psi \rangle = 1 \end{aligned} \quad (75)$$

This result follows from the expansion rule

$$\int | q' \rangle \langle q' | dq' = I = \text{unit-matrix} \quad (76)$$

which is shown easily by multiplying on both sides with an arbitrary ket $| q'' \rangle$, and using the orthonormal conditions of Equation (72) for the q -variable. Equation (75) shows that the state vector $|\psi\rangle$ is normalized.

A similar approach can be used for the p -variables. Consider the ket-vector above $|\psi\rangle$, expressed in terms of an orthonormal set of $| p' \rangle$ eigenvectors belonging to (eigen) values p' , of an observable p , then the function $\phi(p')$ may be defined as:

$$\phi(p') = \langle \psi | p' \rangle \quad (77)$$

Substitution of Equations (74) and (76) into Equation (62) leads to

$$\begin{aligned} \langle \psi | p' \rangle &= \int \langle \psi | q' \rangle \langle q' | p' \rangle dq' \\ &= \int \langle \psi | q' \rangle e^{2\pi i p' q'} dq' \end{aligned} \quad (78)$$

from which follows the important identification

$$\langle q' | p' \rangle = e^{2\pi i p' q'} \quad (79)$$

The same procedure, substituted into Equation (63), gives

$$\langle p' | q' \rangle = e^{-2\pi i p' q'} \quad (80)$$

as required of the scalar product inversion rule. For a multiple set of independent observables q_1, q_2, \dots, q_n , and corresponding set of p -variables p_1, p_2, \dots, p_n , the following is obtained

$$\begin{aligned} \langle q_1', q_2', \dots, q_n' | p_1', p_2', \dots, p_n' \rangle &= \langle q_1' | p_1' \rangle \langle q_2' | p_2' \rangle \dots \\ \dots \langle q_n' | p_n' \rangle &= e^{2\pi i (p_1' q_1' + p_2' q_2' \dots + p_n' q_n')} = e^{2\pi i (p' \cdot q')} \end{aligned} \quad (81)$$

where the shorthand notation is used:

$$\langle q' | = \langle q_1', q_2', \dots, q_n' | = \langle q_1' | \langle q_2' | \dots \langle q_n' | \quad (82)$$

These relationships are fundamental to the following definition of average:

$$\langle q_A | = \int w(q') \langle q' | dq' \quad (83)$$

Hence,

$$\begin{aligned} \langle q_A | p' \rangle &= \int w(q') \langle q' | p' \rangle dq' \\ &= \int w(q') e^{2\pi i p' \cdot q'} dq' \end{aligned} \quad (84)$$

This is the characteristic function for the distribution function $w(q')$. For the matched case, Equation (84) becomes

$$\begin{aligned} \langle q_A | p' \rangle &\stackrel{C}{=} \int w(q') (1 + 2\pi i p' \cdot q') dq' \\ &\stackrel{C}{=} e^{2\pi i p' \cdot q_A} \end{aligned} \quad (85)$$

which is the same as Equation (66). Hence, for the cognitive limit, $\langle q_A | p' \rangle$ agrees with Equation (79). Equation (85) describes the basic process of measurement of parameters p' . What

one finds from Equation (84) is that this process is simply given by an average sensor state $\langle q_A |$ "receiving" the parameters p' through a scalar product operation with the corresponding state vector $| p' \rangle$. Another useful form may be derived from Equation (84).

$$\langle q_A | p' \rangle = \langle \psi | B(p') | \psi \rangle = \langle B(p') \rangle \quad (86)$$

where

$$B(p') = \int | q' \rangle \langle q' | e^{2\pi i p' \cdot q'} dq' \quad (87)$$

The operator $B(p')$ has the additive property

$$B(p' + p'') = B(p') B(p'') \quad (88)$$

which is easily verified upon substitution into Equation (24) and using the orthonormal properties of $| q' \rangle$ vectors.

On the right-hand side of Equation (86), the notation for average value for the observable, often used in quantum theory, is observed.

Notice that the averaging process always is initiated with the introduction of the observer's intention state $| \psi \rangle$. If ψ does not appear in the equations, as in Equation (79) or (87), a structural relationship between operations is said to exist. When ψ does appear, it is called an intentional relationship, because the observer's intention, or attention field, is represented.

From Equation (86), it follows that $B(p')$ may be considered as the structural operator which initiates the "measurement of p' ". The measurement itself is the result of an intentional average over this operator.

Another interesting result, which follows from Equation (83) is

$$\langle q_A | q'' \rangle = w_A(q'') \quad (89)$$

which shows the probability density derived from the state. If $\langle q_A |$ were an eigenvector $\langle q'' |$, instead of an average of these, the familiar result $\delta(q'' - q')$ in Equation (89) would be obtained. This shows that introduction of the state $\langle q_A |$ causes a spreading of the distribution function. In contrast to $\langle q'' | q'' \rangle = 1$, $\langle q_A | q_A \rangle$ becomes

$$\begin{aligned} \langle q_A | q_A \rangle &= \int \langle q_A | q' \rangle \langle q' | q_A \rangle dq' \\ &= \int w^2(q') dq' < 1 \end{aligned} \quad (90)$$

Hence, the $\langle q_A |$ vector is not normalized! The following operator will be used often in future work:

$$\begin{aligned} B_A &= | q_A \rangle \langle q_A | = \int w(q') | q' \rangle \langle q_A | dq' \\ &= \int (| q' \rangle \langle q_A |)^2 dq' \end{aligned} \quad (91)$$

The last step follows easily from Equation (89).

A further useful development is to substitute a uniform distribution instead of $w(q')$ in Equation (83). The following can now be defined:

$$| 0 \rangle = \int | q' \rangle dq' \quad (92)$$

Hence

$$\langle p' | 0 \rangle = \int \langle p' | q' \rangle dq' = \delta(p') \quad (93)$$

and

$$\langle q' | 0 \rangle = \int \langle q' | q'' \rangle dq'' = 1 \quad (94)$$

The state $| 0 \rangle$ thus acts as a kind of ground state for the q -variables. In order to simplify notation, $\langle 0 |$ is not allowed to represent the bra-vector corresponding to the ket- $| 0 \rangle$ for the q -variables, as would usually be the case; instead, the $\langle 0 |$ notation is preserved for the corresponding state in the p variables!

$$\langle 0 | = \int \langle p' | dp' \quad (95)$$

By using this (one-sided) convention for the null-vector, a convenient representation scheme is developed. Hence

$$\langle q_A | 0 \rangle = \int \langle q_A | q' \rangle dq' = 1 \quad (96)$$

but

$$\begin{aligned} \langle 0 | q_A \rangle &= \int \langle p' | q_A \rangle dp' = \int \int w(q') e^{-2\pi i p' \cdot q'} dq' dp' \\ &= \int w(q') \delta(q') dq' = w_A(0) \end{aligned} \quad (97)$$

Similarly

$$\langle 0 | 0 \rangle = 1 \quad (98)$$

If two operational definitions lead to the same cognitive-state values, they are equivalent, although not equal operations. For instance, it is easy to show that $B_A = | q_A \times q_A |$ in Equation (91) is equivalent to

$$B'_A = | q \rangle \langle q |_A = \int w(q') | q' \rangle \langle q' | dq', \quad (99)$$

because

$$\langle p_1 | q_A \rangle \langle q_A | p_T \rangle \stackrel{C}{=} \exp [2\pi i (p_T - p_1) \cdot q_A] \quad (100)$$

and also

$$\langle p_1 | \int w(q') | q' \rangle \langle q' | dq' | p_T \rangle \stackrel{C}{=} \exp [2\pi i (p_T - p_1) \cdot q_A] \quad (101)$$

for all values of p_T and p_1 . The definition $B_A = | q_A \rangle \langle q_A |$ gives more information for the well-matching condition under detection. This can be seen as follows. First, evaluate $\langle q_A | p_T \rangle$ which produces the "received signal" as usual

$$\langle q_A | p_T \rangle \stackrel{C}{=} \exp [2\pi i p_T \cdot q_A] \quad (102)$$

and hence the well-matched condition is $p_T \perp q_A$. Next, the following is obtained:

$$\langle p_1 | q_A \rangle \langle q_A | p_T \rangle \stackrel{C}{=} \exp [2\pi i (p_T - p_1) \cdot q_A] \quad (103)$$

and this leads to the well-matching condition $p_T = p_1$.

If one would follow through with $B'_A = | q \times q |_A$, one would find the same form

$$\langle p_1 | B_A | p_T \rangle \stackrel{C}{=} \exp [2\pi i (p_T - p_1) \cdot q_A] \quad (104)$$

which is well matched if $p_T = p_1$. Hence, the operator $B_A = | q_A \times q_A |$ gives more information, because it also requires that $p_T \cdot q_A = 0$. In some cases when this distinction is understood, $B_A \approx B'_A$ and, hence,

$$B_A = | q_A \rangle \langle q_A | \approx | q \rangle \langle q |_A = \int w(q') | q' \rangle \langle q' | dq' \quad (105)$$

This observation is important because, if $w(q')$ is replaced by a constant, the operator $\int w(q') | q' \rangle \langle q' | dq'$ becomes the identity

operator I (see Equation (77)). Thus, Equation (105) may be viewed as an extension of this identity operator for finite variances of the distribution function.

By using a similar argument, the following can be defined:

$$M_A = | p_A \rangle \langle p_A | = | p \rangle \langle p |_A = \int w(p') | p' \rangle \langle p' | dp' \quad (106)$$

This definition will be found later to be indeed the most correct form to describe the sorting operation.

Cognitive Operations: An Introduction

The notational developments of the previous subsection are now put to use to describe cognitive processes. These are the processes which lead to cognitive states, for which the \mathbb{E} symbolism was assigned.

Essentially all these processes involve the maximization of an integral, like in matched filtering processes. For detection, this amounts to maximizing the received signal from the target, after the signal has passed through a matched filter receiver. For our purposes, only Gaussian noise sources were considered, which result in the autocorrelation function for the attention field $\phi(p)$ to be received. The cognitive process consists of finding the conditions when this function is a maximum. This is called the matched condition, i.e., the observer's attention field, or expectation-field, is now matched to the target. In other words, the target parameters have been received and measured through the optimum filtering procedure. All this was shown to be incorporated by the formula

$$\langle q_A | p_T \rangle = \int w(q') e^{2\pi i p_T \cdot q'} dq' \quad (107)$$

where the target is represented by the parameters: $p_T = (p_1, p_2, \dots, p_n)$. For the matched condition it was found that

$$\langle q_A | p_T \rangle \stackrel{c}{=} e^{2\pi i p_T \cdot q_A} \quad (108)$$

In other words, the maximum strength of the signal received is normalized to $\sigma = 1$. If also the phase-term in Equation (108) is zero, the target is said to be well-matched. Not all cases of matching result in well-matched conditions. However, for cognition to be complete, a well-matching condition is required. For the detection case, it was found in an earlier section that target-matching also results in well-matching.

As shall be seen, there are essentially two basic types of cognitive processes. The first is called detection which was discussed above, and it involves an operation in the space of q -variables (the sensors). The second operation is called sorting, and it involves an operation of a quite different nature in the space of p -variables. The cognitive process, as it evolves in time, consists of a set of sequential operations: detection, sorting, detection, sorting, etc. This process follows the sequence of body-sensors B and mind M operations in order from right to left:

$$\dots M B M B = \dots A A = A \quad (109)$$

At each stage between M and B , a well-matching condition must exist. This sequence also has a quantum mechanical analogue which describes the Feynman path integral time-evolution of a dynamical system. The quantum mechanical description consists of the following sequences:

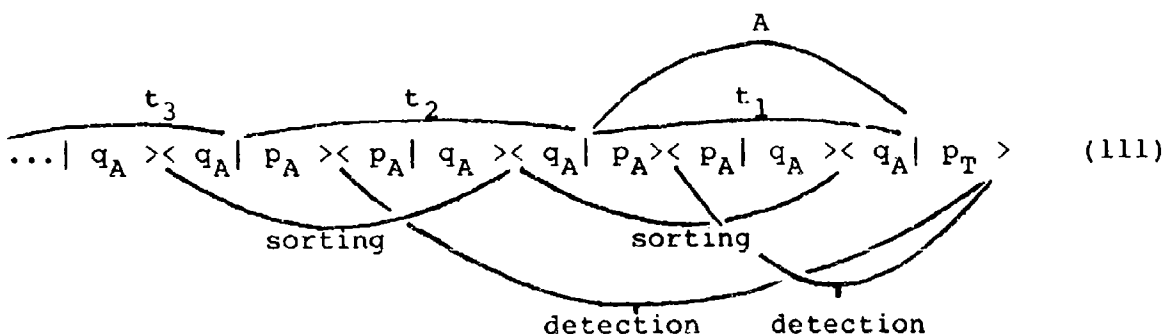
$$\overbrace{\dots | q_3 \rangle \langle q_3 |}^{t_3} \overbrace{| p_2 \rangle \langle p_2 | q_2 \rangle \langle q_2 |}^{t_2} \overbrace{| p_1 \rangle \langle p_1 | q_1 \rangle \langle q_1 |}^{t_1} | p_T \rangle \quad (110)$$

where each index depicts a new time interval for the variable p or q , starting with $t = t_1$ on the right-hand side, and $t = t_3$ on the

left-hand side. On the far right is shown the target, represented by state vector $|p_T\rangle$ which is exterior to the system.

Notice that the sequence in Equation (110) is free of the observer's $|\psi\rangle$ intentional state and, hence, is a structural sequence, whereas Equation (109) includes the observer's intention. In order to arrive at Equation (109), starting from the sequence in Equation (110), one must introduce some type of averaging process.

It will be shown that a sequence of operations of the following type will be obtained from the sequence given in Equation (110):



After one detection and one sorting operation is completed, the process repeats itself with $\langle q_A|$ internally observing $|p_T\rangle$ externally, as indicated by the arrow below the sequence.* The sequence of operations thus marches as it were to the right after each matching operation is completed. A similar process is familiar in quantum theory, based upon the sequence in Equation (110), where the "marching to the right" is achieved by integrations dq_1 , dp_1 etc. For example, $\langle p_1|$ moves to the right:

* Later on we find an exception to this rule; $\langle q_A|$ can belong also to the sorting process.

$\int \langle p_1 | q_1 \rangle \langle q_1 | p_T \rangle dq_1 = \langle p_1 | p_T \rangle$ (112)
 by using the identity expansion rule. In our case, the procedures and their physical significance are quite different, being based on cognitive principles, but the idea of 'marching to the right' in time is similar.

First, try to understand the sequence of operations in Equation (111). The operations which occur in Equation (111) are the scalar products $\langle p_A | q_A \rangle$. One has to distinguish between operations which occur within one time frame and those that occur between overlapping time frames. The latter refer to the so-called dynamic case. For the static case,

$$\langle p_A | q_A \rangle = \int w(p') \langle p' | q_A \rangle dp' \quad \text{and} \quad \int w(p') e^{-2\pi i p' \cdot q_A} dp' \\ \quad \text{and} \quad \int e^{-2\pi i p_A \cdot q_A} \quad (113)$$

Hence, in the static case, the condition for well-matching between mind and body-sensor operations is that $p_A \cdot q_A = 0$ such that p_A and q_A are at all times orthogonal.

This condition is familiar to radar engineers, where the antenna system A is constructed such that at all times the maximum direction p_A of the main beam $\phi(p)$, which attempts to locate the target at p_T , is orthogonal to the antenna aperture space q , which includes the vector q_A . This, in turn, gives the position of maximum intensity of illumination function $\psi(q)$ on the aperture. The dynamic case, referred to above, is more complicated and will be discussed later with the sorting operation.

Continuing our discussion for the static condition, it seems tempting to associate mind with

$$M_A = | p_A \rangle \langle p_A | \quad (114)$$

and body with

$$B_A = | q_A \rangle \langle q_A | \quad (115)$$

in Equation (111). For the case of well-matching, $\langle p_A | q_A \rangle \stackrel{C}{=} 1$, and the following can be written:

$$A = | p_A \rangle \langle p_A | q_A \rangle \langle q_A | \stackrel{C}{=} | p_A \rangle \langle q_A | \quad (116)$$

It is then easy to show that $A A = A$ for the matched condition and hence the basic law for cognition is satisfied. However, our enthusiasm after this result is tempered somewhat because

$$M_A M_A = \langle p_A | p_A \rangle M_A \quad (117)$$

and, as was found previously, $| p_A \rangle$ is not normalized, i.e.,

$$\langle p_A | p_A \rangle = \int w(p')^2 dp' < 1 \quad (118)$$

The same holds true for B_A . Thus, M_A and B_A as defined above are not of the cognitive type. An even more serious objection against relating B_A and M_A to the classical "body" and "mind" operations of cognitive theory is that M_A contains the state-vector $| p_A \rangle$ twice, once as a bra- and then as a ket-vector, whereas the classical mind-operator $M = \hat{v} \times \hat{v} + p_A \times \hat{u}$ contains the p_A symbol only once and then as a ket-vector. Similar arguments apply to B_A . One way out of this dilemma is to consider the form of Equation (116) on the left only as a mathematically convenient way of writing A initially, and because $\langle p_A | q_A \rangle \stackrel{C}{=} 1$ the duplication of symbols in the matching process is eliminated.

Another procedure would be to introduce, instead of Equation (116), the form

$$A = M B = | p_A \rangle \langle 0 | 0 \rangle \langle q_A | = | p_A \rangle \langle q_A | \quad (119)$$

Now a clear-cut, almost one-to-one, representation with the cognitive theory is obtained. But, it turns out Equation (119) is

not as useful as Equation (116), because Equation (119) already describes A in the cognitive state, whereas Equation (116) on the left describes the process which guides us to this state.

This discussion leads one to consider Equation (116) as indeed the correct mathematical representation of the processes which produce the cognitive states, although M_A and B_A are not direct extensions of the classical mind and body states. This, however, should come as no surprise, because one is dealing here with pre-cognitive processes which only after the matching requirement become cognitive states.

The matching procedure consists of maximizing a 'received signal' strength σ , to a maximum value $\sigma = 1$. Once this is done, one can properly speak of a cognitive state, where the target, or object, has been detected within a small margin of uncertainty. Finally, the well-matched condition is a process which takes place within the cognitive framework such that now the target is accurately pinpointed, or recognized, and the internal framework p_A replicates the target's p_T . In other words, the target's characteristics have been recognized and interpreted by the observer's mind process and a replica of the external phenomenon has been created in the space p of the observer's attention field $\phi(p)$.

These preliminary remarks are not to be construed as definitive. In the following sections some of the definitions of "mind" and "body" operation as introduced above will have to be refined and modified to suit our needs and for a more precise delineation.

The Detection Process

The detection process is the simplest to understand and to analyze because of the radar-analogue which was discussed in detail before. Consider the antenna with attention field $\phi(p)$ trying to lock on to the target. The maximum beam strength is pinpointed in direction p_A . The target manifests itself by its "location" parameters $p_T = (p_1, p_2, \dots, p_n)$. However, as was found before,

only the direction of target location can be measured. Hence, instead of p_T only \hat{p}_T , the unit vector of p_T ,* can be measured.

This remark gives a clue to the detection process, and it also opens up the need for further processing, which was called sorting, and which is discussed later. Whereas sorting is a strictly dynamic process and has no static equivalent, the detection process works only under static conditions. This is because the internal locking operation $p_A \cdot q_A = 0$ or $p_A \perp q_A$ is built internally into the system and is independent of where the target is located. The vector q_A locates the maximum field strength on the antenna aperture, which carries the space of q-variables, and thus q_A also is located in the antenna aperture space. Hence the condition $p_A \cdot q_A = 0$ also points out that the search for the target is done outside of the q-variable, aperture, space. This is characteristic for a remote sensing mechanism. Connected with the remote sensing aspect of detection is the fact, noted above, that only the direction \hat{p}_T of the target can be measured.* Hence, the p-space of the target has an extra dimension: the distance or range of the target. As will be seen shortly, it is the remote sensing aspect of detection which makes the target matching operation possible. If the spaces of p- and q-variables coincided, no detection of target features would be possible.

Still another way of looking at it is to consider the antenna aperture q space as generating a wavefront, of the attention field, which propagates to or from the target. For the detection process, one is not concerned with the movement of the wave front itself which is a dynamic process and which, of course, also depends on the movement of the target as the antenna is locked on and follows the target motion in p-space. The detection process deals strictly with the locking-on operation, i.e., the direction p_A of the aperture, q-variable plane, and not with the propagation aspects of the wave front represented by the aperture plane. The "propagation

*Later on this condition will be relaxed such that p_T is no longer restricted to a unit vector.

constant", which fixes the length of vector p_A , is determined by the sorting process.

Having thus made clear the distinction between detection and sorting, we now proceed to analyze how the detection process operates. The so-called received signal from the target, based on an expectation function $\phi(p)$ and corresponding $\psi(q)$, is given by the characteristic function

$$\langle q_A | p_T \rangle = \int w(q) e^{2\pi i p_T \cdot q} dq \quad (120)$$

Here, $w(q) = |\psi(q)|^2$ is the target estimation function; it represents the observer's initial guess where the target might be located. Usually, the wider the distribution function $w(q)$, the more sensors activated and the sharper the corresponding target beam function $|\phi(p)|^2$ will be. If one substitutes a Gaussian estimation function for $w(q)$, one finds

$$\langle q_A | p_T \rangle = \frac{1}{|\sqrt{2\pi} \sigma|} \int e^{-\frac{\sigma^{-2}}{2} (q - q_A)^2} e^{2\pi i p_T \cdot q} dq \quad (121)$$

Some notational conventions have to be cleared up. The vector q consists of n components $q = (q_1, q_2, \dots, q_n)$ which define the q -space. Similarly, the target vector p_T has $(n+1)$ components. The extra dimension of the p -space accounts for "remote sensing", i.e., the target is considered to be located at some large distance apart from the observer's "body-sensor" system of data gathering devices. In that sense, the target is "remote" from the q -space sensory mechanism. In radar-terminology, the target is located in the far-field of the antenna system.

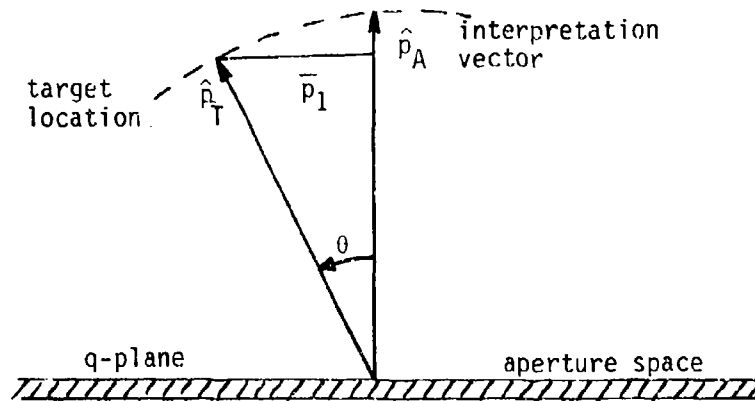


Figure 4. Remote Viewing Geometry

Figure 4 illustrates the remote viewing geometry. For the scalar product, one now derives the important relationship

$$\hat{p}_T \cdot q = \bar{p}_1 \cdot q = \hat{p}_1 \cdot q \sin \theta \quad (122)$$

where \bar{p}_1 is the part of \hat{p}_T which is parallel to the q-space. The integration of Equation (121) can now be performed with all vectors located in the q-space. The symbol σ in the demoninator of the exponent stands for a diagonal matrix containing the variances σ_i , while $|\sqrt{2\pi} \sigma|$ in front of the integral is the symbolic notation for the determinant of the 2×2 matrix $\sqrt{2\pi} \sigma$, i.e., $|\sqrt{2\pi} \sigma| = \det(\sqrt{2\pi} \sigma)$. Hence, $\frac{\sigma}{2} (q - q_A)^2$ is a symbolic notation for

$$\frac{1}{2} \begin{bmatrix} 1/\sigma_1^2 & & & \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ & & & 1/\sigma_n^2 \end{bmatrix} \begin{bmatrix} q_1 - q_{1A} \\ q_2 - q_{2A} \\ \vdots \\ q_n - q_{nA} \end{bmatrix} \cdot \begin{bmatrix} q_1 - q_{1A} \\ q_2 - q_{2A} \\ \vdots \\ q_n - q_{nA} \end{bmatrix} \quad (123)$$

With these provisions, the received signal Equation (121) is evaluated as follows:

$$\langle q_A | p_T \rangle = e^{-\frac{\sigma^2}{2} \hat{p}_1 \cdot \hat{p}_1 \sin^2 \theta} e^{2\pi i \hat{p}_1 \cdot q_A \sin \theta} \quad (124)$$

The matching condition is now apparent. The amplitude-signal part in Equation (124) becomes a maximum signal of unity if $\theta = 0$, in which case, p_A points in the direction of the target \hat{p}_T . The phase part then also becomes zero, i.e., the matching condition also produces well-matching. The matching condition also gives insight as to how the detection operation works. The angle θ is made to approach zero by tilting the whole q -space or antenna-aperture plane while keeping the beam-function $\phi(p)$ fixed to this space. By doing this, one achieves that $p_A \cdot q_A = 0$ is satisfied under all conditions. If the target is found, $\theta = 0$ and p_A points in the target direction, i.e., $p_A = p_T$.

This technique for measuring a set of target parameters (p_1, p_2, \dots, p_n) is different from a sequential set of independent measurements $p_1 q_1, p_2 q_2, \dots, p_n q_n$. Here, the sum is measured as the total sum phase $(p \cdot q)$ and the individual components are resolved by the locked-in condition $p_A \cdot q_A = 0$. Also notice that the variances σ_i can be chosen almost arbitrarily. Obviously, the larger one chooses each σ_i , the sharper the beam-function $\phi(p)$ will be, which results in a correspondingly more accurate target parameter measurement.

The matching process described by $\theta \rightarrow 0$ actually consists of two separate procedures which are called matching and well-matching. There are two ways to describe this. The first is that

of optimizing the signal, i.e., target-matching is done through the limiting condition $\sin^2 \theta \rightarrow 0$. In other words, matching comes first, after which one has well-matching.

Another way of explaining these processes is to observe that as θ becomes small, one can write for Equation (120) in the exponent

$$\begin{aligned} \langle q_A | p_T \rangle &= \int w(q) (1 + 2\pi i \hat{p}_T q \sin \theta) dq = \\ &= (1 + 2\pi i \hat{p}_T q_A \sin \theta) \approx e^{2\pi i \hat{p}_T q_A \sin \theta} \end{aligned} \quad (125)$$

Hence, the matching condition has been satisfied; only well-matching (making the phase equal to zero) is to be accomplished. This shows there are indeed two independent processes at work.

In summary, the detection process was shown to consist of the permanent condition $p_A \cdot q_A = 0$, or $\langle p'_A | q_A \rangle = 1$, and the matching and well-matching condition $\langle q_A | p_T \rangle = 1$. Combined, these processes amount to the condition

$$\langle p'_A | q_A \rangle \langle q_A | p_T \rangle = 1 \quad (126)$$

This was called the detection sequence in a previous section. Now a prime was added to p_A in Equation (126) to indicate that p'_A in this case does not indicate an averaging of states $\langle p' |$, but merely a labeling of a single state with value p'_A which corresponds to the maximum beam function $\phi(p)$. Hence, the process of Equation (126) can be rewritten as

$$\langle p'_A | p_T \rangle = \delta(p'_A - p_T) \quad (127)$$

which now agrees with the matched condition of Equation (126) if $p'_A = p_T$. By comparing Equations (126) and (127), one has an example of the "marching to the right" principle mentioned in a previous subsection. The integration over q -space amounts to

replacing $B_A = |q_A \times q_A|$ in Equation (127) by the unit matrix, which moves $\langle p_A |$ to the right next to $| p_T \rangle$.

Notice that the "remote sensing" requirement that the target vector p_T is a unit vector does not play an essential role in the equations for detection. Hence, one may relax this condition for the general case and assume that target parameters $p_T = (p_1, p_2, \dots, p_n)$ are measured without the unit vector normalization. This generalization removes an undue restriction on the internal representation p_A which is used for the sorting operation. The details of the sorting process are discussed in the following subsections.

Introduction to the Sorting Process

The sorting process is in many ways completely different from detection of the target. Sorting is an internal dynamic process operating in p-space after detection as an operation on q-space is completed in one time period. One may wonder what else must be done after target parameters have been measured.

What comes after data gathering is usually called data processing. Hence 'sorting' may be identified as some type of data processing. Whereas data processing consists of almost any operation on the data, such as taking averages or obtaining correlations, the sorting process will be shown to have a very specific purpose: to identify the target as a true object! This may seem like an ambitious project strewn with conceptual difficulties, but one can show that there is a systematic path to be followed towards this objective. What makes object-construction seem difficult is that a subjective element is involved with making the decision that something is an object.

Clearly, some analytical principles have to be developed first before much headway or understanding towards solving this problem in perception can be expected to take place. One must strike a balance between the mathematical presentation and the physical interpretation at each step of the development. As is usually the

case, the most difficult problems are conceptual in nature, and here one is dealing with the very nature of what "being a concept" for an object means! Hence, progress will be slow at first and systematic. On the mathematical side, there will be the definition of what constitutes an object and its connection with the sorting process. Ideas related to memory and experience will provide the subjective basis for these developments.

The sorting process itself has an analog with the Feynman path integral time-development of an elementary free particle. This was described previously as the process

$$\langle q_2 | p_1 \rangle \langle p_1 | q_1 \rangle \quad (128)$$

where the indices relate to variables in different time frames, i.e., $|q_1\rangle = |q_1, t_1\rangle$, etc. Notice that there is some ambiguity in the notation used in Equation (128), because there is no explicit presentation of time, as the system evolves. The classical way to introduce the development in time is through the time-evolution operator

$$\langle q_2 | p_1 \rangle = \langle q_2 | e^{-2\pi i \epsilon H} | p_1 \rangle \quad (129)$$

where $\epsilon = t_2 - t_1$ denotes the time change between the two states, and H is the classical quantum mechanical Hamiltonian operator which stands for total energy of the system. For an infinitesimal time interval, one obtains from Equation (129) the regular approximation*

$$\langle q_2 | p_1 \rangle = e^{2\pi i p_1 \cdot q_2} e^{-2\pi i \epsilon h(p_1, q_2)} \quad (130)$$

* See Reference (13), p. 432.

For the classical free-particle with mass m , one has

$$h(p_1, q_2) = \frac{p_1^2}{2m} + V(q_2) \quad (131)$$

where on the left side is the total energy and on the right are the contributions from kinetic energy and potential $V(q_2)$. The theory can be developed based on these relationships, as was done in Feynman's original work*. For our purposes, however, one rather looks upon Equation (131) as an approximation of the relativistic form which is valid only for small velocities, i.e.,

$$h = c p_1^c = c \sqrt{m^2 c^2 + p_1^2} \approx mc^2 + \frac{p_1^2}{2m} \quad (132)$$

where c is the velocity of light and p_1^c is the relativistic form for total energy (actually $h = c p_1^c$ is energy).** In the following part, as in Equation (132), one neglects the effect of potential $V(q_2)$ on the dynamics of the free particle*** such that the equation for energy in Equation (132) becomes

$$p_1^{c^2} = p_m^2 + p_1^2 \quad (133)$$

where $p_m = m c$. If now one substitutes Equation (132) into Equation (130), one gets a curious relationship for Equation (128):

$$\langle q_2 | p_1 \rangle \langle p_1 | q_1 \rangle = e^{2\pi i (q_2 - q_1) \cdot p_1 - 2\pi i \epsilon c p_1^c} \quad (134)$$

Notice that the dynamic time-change $\mu^c = \epsilon c$ not only defines a change in q -state by $\mu = q_2 - q_1$ but also induces a new parameter p_1^c ! One could also have written for Equation (134):

* See Reference (14).

** See Reference (15), page 118, equation (23) for derivation.

*** The effect of potential will be introduced later.

$$\begin{aligned}
\langle q_2, ict_2 | p_1, ip_1^C \rangle & \langle p_1, ip_1^C | q_1, ict_1 \rangle \\
& = e^{2\pi i \mu \cdot p_1 - 2\pi i \mu^C p_1^C}
\end{aligned}
\tag{135}$$

Notice the introduction of complex parameters, familiar in special relativity, although the introduction of the same i on the left- and the right-hand sides of Equation (135) should be used with some caution. For our purposes, one may consider Equations (133) and (134) as the basic equations related to the system.

The cognitive sorting process is described as an event in p -space by the mind-operator M_A which is now defined as follows:

$$\langle q_2 | M_A | q_1 \rangle = \int w(p) \langle q_2 | p \rangle \langle p | q_1 \rangle dp \tag{136}$$

After substitution of Equation (135), this becomes

$$\langle q_2 | M_A | q_1 \rangle = \int w(p) e^{-2\pi i (\mu^C p^C - \mu \cdot p)} dp \tag{137}$$

where primed subscripts were omitted in the integrand. One notices at once the difference between the sorting process and detection of the target. An extra parameter, defined by Equation (133), has been introduced, which is due strictly to the dynamic change that has taken place.

Our derivation of Equation (137) was based on the quantum-mechanical analogue. It will be of foremost importance to assign to Equation (133) a more fundamental significance, which relates to our objective: to define the object in terms of the known measured data, by some type of sorting operation which leads to target-identification.

The key to this development is Equation (133) which introduces two new parameters, p^C and p_m , that are internally related to p by Equation (133). In classical dynamics these parameters stand for

the total energy and the rest-mass energy of the elementary object. It is clear that substantial progress can be made if one can attach new and more general significance to these classical parameters. These questions will be left for a later discussion.

At this moment, one can evaluate integral Equation (137) based on methods introduced with the detection procedure. One first looks for conditions for which the phase in the integrand of Equation (137) becomes stationary. In classical mechanics this condition defines the equations of motion for the elementary particle.

Another way of looking at this condition is to observe that one is searching for conditions that define a stationary phase-front in p-space at each moment in time. As time progresses, the description becomes that of a moving wave, its phase-front moving in time through p-space. The particle or object motion is then associated with a set of trajectories or rays which are orthogonal to the stationary phase-front. The object or particle is then said to move along one of the ray-path trajectories in p-space according to the laws of motion which define the stationary phase-fronts and which, by the same token, also define the object. The only thing still missing from this process of object formation and identification is to justify the introduction of p_m as the "mass-term" of the object. The analysis of this question will be postponed until a later subsection.

In summary, the sorting process thus turns out to be the search for the definition of the object, if its basic parameters in time are known from the detection process. Hence, sorting is equivalent to target identification.

From detection, one knows what the target data are, but not how the data relate to the target as an object. The fact that a dynamic process is required for target identification is significant because it suggests a statistical averaging scheme as the basic process by which one comes to "know" objects. In other

words, one may expect that experience and memory will play a substantive, subjective, and supporting role in the process of target identification.

SECTION VI

EVALUATION OF SORTING INTEGRALS

The sorting process in cognition is essentially an attempt at target-formation and identification using a given set of target parameter data. The process is described by a so-called sorting integral. The aim is to find the conditions under which the integral is a maximum. These are called the target sorting matching conditions.

The situation is analogous to optimizing the action-integral in classical mechanics. The matching conditions there lead to the equations of motion for the physical state under consideration. The sorting integral resembles a type familiar in quantum theory. There they are called Feynman path integrals. Sorting integrals are more general because they contain a probability density, whereas path integrals apply only for a uniform density. All this will become more apparent as the work progresses.

The One-Dimensional Sorting Integral

In this part, a Gaussian probability density $w(p)$ will be considered. The simplest case is for the target parameter variable p to be one-dimensional. The sorting integral takes on the following form:

$$I_s = \int \frac{e^{-\frac{1}{2\sigma^2}(p - p_A)^2 - 2\pi i (\frac{\lambda}{2} p^2 - \mu p)}}{\sqrt{2\pi} \sigma} dp \quad (138)$$

The integral appears to have four constants: σ is the standard deviation of the Gaussian density, p_A is the average value, λ is a non-negative constant which contains the time-change, whereas μ relates to sensor-changes. The integration ranges over

all values of parameter space p . The integral in Equation (138) has a closed form solution:

$$I_S = e^{-\frac{\sigma^2}{2} (\lambda p_A - \mu)^2} e^{-2\pi i \left(\frac{\lambda}{2} \langle p^2 \rangle - \mu p_A \right)} \quad (139)$$

This result is highly significant, as will be found shortly. Notice that the solution of Equation (139) consists of two exponential terms. The first term, called "signal", is real and it reaches a maximum of "one" when $\mu = \lambda p_A$. The second term is a phase term. It is called the "phase residue." Notice that the phase residue in Equation (139) is simply the phase term in Equation (138) with averaged values in the exponent.

Also observe that the matching condition $\mu = \lambda p_A$ also is found from

$$\frac{d}{dp} \left(\frac{\lambda}{2} p^2 - \mu p \right)_A = 0, \quad (140)$$

i.e., the phase term of the integrand in Equation (138) is stationary at the point p_A . This last condition is useful, because it can be applied as matching condition also when the distribution function is not Gaussian.

Also notice that the standard deviation σ does not enter into the matching condition. If σ is large, the signal in Equation (139) falls off rapidly from the maximum, such that it is important that $\mu = \lambda p_A$ applies strictly for locating the maximum. If σ is small, a broad maximum for the signal is observed; for this case, it is not as essential that $\mu = \lambda p_A$ is strictly satisfied. These observations are important for later developments.

Next, the results are generalized. Consider the one-dimensional sorting integral

$$I_S = \int w_A(p) e^{-2\pi i f(p)} dp \quad (141)$$

where $w_A(p)$ is a general density with average value p_A and $f(p)$ is a general phase term. The interest is mainly in contributions to the sorting integral in Equation (141) which come from the neighborhood of $p = p_A$. Hence, the phase term can be written

$$f(p) = f_A + f'_A (p - p_A) + \frac{f''_A}{2} (p - p_A)^2 + \dots \quad (142)$$

The matching condition requirement, when the signal is maximum, amounts to the stationary phase condition $f'_A = 0$, for $f(p)$. Hence, if the signal is matched, Equation (142) can be written as

$$f(p) = f_A + \frac{\lambda}{2} (p - p_A)^2 + \dots \quad (143)$$

where $f''_A = \lambda > 0$

Substituting Equation (143) into Equation (141), we obtain a sorting integral of the form

$$I_s = \int w_A(p) e^{-2\pi i \frac{\lambda}{2} (p - p_A)^2 + \dots} dp e^{-2\pi i f_A} \quad (144)$$

And because $(p - p_A)$ is small in the neighborhood of $p = p_A$, the exponential phase term can be expanded and integrated as follows:

$$\begin{aligned} I_s &= \int w_A(p) [1 - 2\pi i \frac{\lambda}{2} (p - p_A)^2 + \dots] dp e^{-2\pi i f_A} = \\ &= [1 - 2\pi i \frac{\lambda}{2} \langle (p - p_A)^2 \rangle + \dots] e^{-2\pi i f_A} = \\ &\approx e^{-2\pi i (f_A + \frac{\lambda}{2} \langle (p - p_A)^2 \rangle + \dots)} \approx e^{-2\pi i \langle f(p) \rangle_A} \quad (145) \end{aligned}$$

This result agrees, with good accuracy, with Equation (139) for the special case of a Gaussian distribution function. Hence, the general rule is the "residue", after matching, is the phase term of the integrand in Equation (141) with averaged values in the exponent.

Next, the "residue-averaging" rule is investigated to see if it also applies for non-matched conditions. Start with a phase function expansion around a point p_B as follows:

$$f(p) = f_B + f'_B (p - p_B) + f''_B (p - p_B)^2 + \dots \quad (146)$$

The condition is imposed that the phase-function is stationary, $f'_B = 0$, such that

$$f(p) = f_B + \frac{\lambda}{2} (p - p_B)^2 + \dots \quad (147)$$

where, as before, $f''_B = \lambda > 0$. The previous argument will be made that only values close to the stationary point will contribute substantially to the sorting integral, and that a power series expansion of the phase function Equation (147), as was done in Equation (145), also is applicable to this case. For the non-matched case, however, a factor appears for the signal, which for the matched case was "one." Hence, the series expansion in Equation (145) of the exponential is not directly applicable here.

The signal and phase residue are easily computed if Equation (147) is substituted into Equation (138) and the result is used in Equation (139):

$$I_s = e^{-\frac{\sigma^2}{2} \lambda (p_A - p_B)^2} e^{-2\pi i \langle f(p) \rangle_A} \quad (148)$$

This result, of course, agrees with Equation (145) if $p_A = p_B$, in which case the signal is matched. Therefore, the "residue exponent averaging" rule also holds for the non-matched case.

Equation (148) is very useful in evaluating the sorting integral for a general phase function $f(p)$. The procedure is to select a desirable "object-point" p_B , expand in a power series around that point, impose the stationary rule at p_B , and use the result in Equation (148). In the end, the matching requirement

forces the relationship $p_A = p_B$. It thus seems that the initial freedom of choice of p_B as a desirable object-point is done away with in the end. This is true for the one-dimensional case, but different options are available for the N-dimensional cases.

Even in the one-dimensional case, it is not strictly necessary to impose the condition $f'_B = 0$. If the first derivative is left in the series expression of Equation (146) and Equation (145), Equation (148) becomes

$$I_S = e^{-\frac{\sigma^2}{2} [\lambda (p_A - p_B) - \mu_B]^2} e^{-2\pi i \langle f(p) \rangle_A} \quad (149)$$

The matching condition is now $\lambda (p_A - p_B) = \mu_B$ which suggests that indeed p_B may be chosen arbitrarily. In fact, the deviation of p_B from p_A is measured by the first derivative $\mu_B = -f'_B$. It appears, however, that the last condition has less practical interest. The latter case may be translated into the condition $\mu_A = 0$ which has real physical significance because it specifies that the stationary point of $f(p)$ is located at p_A . This is the original case. The condition $\frac{df}{dp} = 0$ at p_A is central to the understanding of the sorting process. The following sections will show how this condition is applied to more general cases.

The Two-Dimensional Sorting-Integral

Consider the case where the parameter variable p consists of two independent measured variables p_1 and p_2 . The sorting integral is, as defined in the previous section,

$$I_S = \int w_A(p_1, p_2) e^{-2\pi i f(p_1, p_2)} dp_1 dp_2 \quad (150)$$

Because p_1 and p_2 are independent, the densities are separable:

$$w_A(p_1, p_2) = w_A(p_1) w_A(p_2) \quad (151)$$

Let the phase-function have the normal form:

$$f(p_1, p_2) = -(\mu_1 p_1 + \mu_2 p_2) + \frac{1}{2} (\lambda_{11} p_1^2 + 2 \lambda_{12} p_1 p_2 + \lambda_{22} p_2^2) \quad (152)$$

For Gaussian densities, Equation (150) can be evaluated based upon the one-dimensional case solution. First integrate over p_2 , which gives a signal term and the exponent of the phase residue $\langle f(p) \rangle_{A2}$. Both signal and phase residue are still functions of p_1 . The matching conditions still do not depend on σ . For p_1 and p_2 , the following matching conditions should be satisfied:

$$\mu_1 = \lambda_{11} p_{A1} + \lambda_{12} p_{A2} \quad (153)$$

$$\mu_2 = \lambda_{22} p_{A2} + \lambda_{12} p_{A1} \quad (154)$$

Now, check if indeed these conditions are met. For the signal after integration with p_2 , find an exponential with exponent

$$- \frac{\sigma_2^2}{2} (\lambda_{22} p_{A2} + \lambda_{12} p_1 - \mu_2)^2 \quad (155)$$

which clearly shows the dependency on p_1 . Now, substitute the Equation (154) into Equation (155) which gives

$$- \frac{\sigma_2^2 \lambda_{12}^2}{2} (p_1 - p_{A1})^2. \quad (156)$$

Combined with the exponent of the density-term for p_1 , the density exponent now becomes

$$- \frac{1}{2} (\sigma_1^{-2} + \sigma_2^2 \lambda_{12}^2) (p_1 - p_{A1})^2 \quad (157)$$

Upon integration with p_1 , the matching condition for p_1 is indeed satisfied by Equation (154). What has been changed by Equation (157) is an adjustment to the standard deviation of p_1 which now becomes

$$(\sigma_1')^2 = \frac{\sigma_1^2}{1 + \frac{\sigma_2^2}{\sigma_1^2} \frac{\lambda_{12}^2}{\lambda_{12}^2}} \quad (158)$$

If the coupling term λ_{12} is small, the adjusted σ_1' approaches σ_1 . Obviously, if the integration had started with p_1 , the roles of indices would have been interchanged in Equation (158), which shows an asymmetry. A more serious objection is that the change of scale due to σ_1' effects the integration, such that the matched condition for the signal is no longer "one". These results clearly show an inconsistency in the above approach, because there should be symmetry in the matching conditions for p_1 and p_2 taken together.

The mistake made above was to consider partial derivatives in one variable as a sufficient procedure, without taking into account boundary conditions of all other variables. The correct matching condition will be shown to be

$$\nabla_p f_A = 0 \quad (159)$$

where

$$\nabla_p = \left(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \dots \right) \quad (160)$$

is the conventional "del"-operator. Equation (159) expresses a stationary condition for $f(p)$ at the point p_A . Applying the operator in Equation (159) to Equation (152), we obtain:

$$\mu_1 = \lambda_{11} p_{A1} + \lambda_{12} p_{A2} \quad (161)$$

$$\mu_2 = \lambda_{22} p_{A2} + \lambda_{12} p_{A1} \quad (162)$$

which agrees with Equations (152) and (153) and which is symmetric in the p variables. Now consider Equation (159) as the physically and mathematically correct condition for target matching in the n -dimensional case. Hence, the sorting integral in Equation (150) indeed obtains the signal "one" for that condition. The residue-phase term then has the exponent $\langle f(p) \rangle_A$, as usual, where the average is taken over both p_1 and p_2 variables.

The evaluation of the sorting integral for general expansions of $f(p_1, p_2)$ follows lines similar to what was done for the one-dimensional case. Details of this will be discussed with the extension to the N -dimensional case which is presented in the following subsection.

The N -Dimensional Sorting Integral

The general sorting integral describes the process by which a "target" is formulated from a given set of observation data p_i . The essential details of how this process is arrived at from general cognition principles and how this works with specific applications are postponed for a later discussion. Of concern here are some elementary integration properties of a so-called sorting integral of the type

$$I_s = \int w_A(p') e^{-2\pi i f(p')} dp' \quad (163)$$

The density function $w_A(p)$ contains the information selected from available data on which our attention is focussed. The phase-function $f(p)$ contains the selection process. Of interest is when $f(p) = \text{constant}$ to describe "phase fronts" which propagate through the n -dimensional variable - p space. If $\nabla f(p) = 0$ for some point p_B , the phase is said to be stationary at that point. The main contribution to the sorting integral may be thought to come from the neighborhood around the

stationary point p_B . Hence, it seems natural to start with a general, vectorial, Taylor expansion of $f(p)$ around p_B ; i.e.,

$$f(p') = f_B + (p' - p_B) \cdot \nabla f_B + \frac{1}{2} ((p' - p_B) \cdot \nabla)^2 f_B + \dots \quad (164)$$

where the standard notation used is

$$\nabla = \left(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \dots, \frac{\partial}{\partial p_n} \right) \quad (165)$$

The label p' was used for the integration variable in the sorting integral in Equation (163), while p is reserved for differentiation.

First bring $f(p)$ to a so-called normal form by introducing new integration variables $p'' = p' - p_B$. The average for p'' now becomes $p''_A - p_B$, and $f(p)$ reduces to normal form

$$f(p'') = f_B + (p'' \cdot \nabla) f_B + \frac{1}{2} (p'' \cdot \nabla)^2 f_B + \dots \quad (166)$$

The differentiation process on $f(p)$ is completed first, before values $p = p_B$ are inserted in Equation (166). The third term is expanded as follows:

$$\begin{aligned} (p'' \cdot \nabla)^2 &= (p''_1 d_1 + p''_2 d_2 + \dots) (p''_1 d_1 + p''_2 d_2 + \dots) \\ &= p''_1^2 d_1^2 + p''_2^2 d_2^2 + \dots + 2p''_1 p''_2 d_1 d_2 + \dots \end{aligned} \quad (167)$$

Here, for short, $d_i = \frac{\partial}{\partial p_i}$, $i = 1, 2, \dots, n$. The following symbolic form will also be needed:

$$\begin{aligned} (p' \cdot \nabla) (p'' \cdot \nabla) &= p'_1 p''_1 d_1^2 + p'_2 p''_2 d_2^2 + \\ &+ (p'_1 p''_2 + p'_2 p''_1) d_1 d_2 + \dots \end{aligned} \quad (168)$$

Hence, in general

$$\begin{aligned} (p' \cdot \nabla) (p'' \cdot \nabla) f_B &= \lambda_{11} p_1' p_1'' + \lambda_{22} p_2' p_2'' + \dots \\ &+ \lambda_{12} (p_1' p_1'' + p_2' p_2'') + \dots \end{aligned} \quad (169)$$

where

$$\lambda_{ij} = \frac{\partial^2 f(p)}{\partial p_i \partial p_j} \Big|_B \quad (170)$$

This identifies the λ_{ij} used previously. One version of the normal form for $f(p)$ in Equation (166) can be expressed as follows:

$$\begin{aligned} f(p'') &= f_B - \mu_B \cdot p'' + \frac{1}{2} (\lambda_{11} p_1''^2 + \lambda_{22} p_2''^2 + \dots \\ &+ 2\lambda_{12} p_1'' p_2'' + \dots) \end{aligned} \quad (171)$$

$$\text{where} \quad \mu_B = - \nabla f_B \quad (172)$$

The matching conditions are now seen as

$$\mu_{Bi} = \lambda_{i1} p_{A1} + \lambda_{i2} p_{A2} + \dots \quad (173)$$

or, in vector notation, remembering that $p_A'' = p_A - p_B$,

$$\mu_B = - \nabla f_B = \nabla (p_A'' \cdot \nabla) f_B \quad (174)$$

The last form is derived from Equation (169).

A second version of the normal form for $f(p)$ is derived from Equation (166), expressed in the variable p' , where

$$p'' = p' - p_B \text{ and } p_A' = p_A :$$

$$f(p') = f_C - \mu_B \cdot p' - (p' \cdot \nabla)(p_B \cdot \nabla) f_B + \frac{1}{2} (p' \cdot \nabla) f_B + \dots \quad (176)$$

where

$$f_0 = f_B + \nu_B \cdot p_B + \frac{1}{2} (p_B \cdot \nabla)^2 f_B \quad (177)$$

The matching conditions now become, remembering that $p'_A = p_A$

$$\nu_B + \nabla (p_B \cdot \nabla) f_B = \nabla (p_A \cdot \nabla) f_B \quad (178)$$

It is easy to verify that Equations (174) and (178) are linearly related through $p_A'' = p_A - p_B$. The full significance of Equation (174) or (178) will gradually become clear in the following subsections when specific examples are discussed.

p_B may be thought of as having some preferred status, such that it seems natural and desirable that the phase function $f(p)$ is stationary for that point; then $\nabla f_B = 0$. From Equations (174) or (178), it then follows that $p_A'' = 0$ and, hence, $p_A = p_B$. This is the natural extension of the one-dimensional case.

For the n -dimensional case ($n \neq 1$), however, other possibilities will open up new avenues for target-decomposition and identification. The basic framework is that under certain conditions $\nabla (p_B \cdot \nabla) f_B = 0$. Then if $\nabla f_B = 0$, it follows that another solution of Equation (174) will be $p_A'' = c p_B$, where c is a constant. Then $p_A = p_A'' + p_B = (c + 1) p_B$, and p_A and p_B are found to be proportionally related. For this case, the freedom of choice of p_B , given p_A , consists of choosing the constant c .

For certain value of c , it is possible to choose the decomposition $p_A = p_B + p_A''$ such that p_B and p_B'' represent "orthogonal objects". This resembles the creation of particles and anti-particles in quantum theory. In cognition, an orthogonal object is that object which is dissimilar to the object under consideration.

If the extra condition $\nabla (p_B \cdot \nabla) f_B = 0$ is not valid, only the value $c = 0$ may be chosen; this restricts the possibilities for target decomposition. Hence, it will be of some consequence

to study the possibility for satisfying the extra condition. These questions will be addressed in the following subsections.

The result of integration of Equation (163) will again produce two exponentials: one is the signal and the other, a phase term, is called the phase residue. After the matching conditions of Equations (173) or (174) are satisfied, the signal becomes "one", and the phase term is the same as in the integrand of Equations (163) with averaged values $\langle f(p') \rangle_A$ in the exponent, as was discussed before.

The significance of the phase residue is that it can be used for memory to classify and recollect the event represented by the sorting process. Details of this will follow.

SECTION VII

APPLICATIONS TO TARGET IDENTIFICATION

Identification of Standard Objects with Euclidean Norm

For this section, we will illustrate the sorting process as it operates on a class of objects p with the most simplest type of norm function p_0 . This case will be the prototype or standard model for more complicated structures. We recall that the sorting operation was found from the dynamic process

$$\langle q_2 | p_1 \times p_1 |_A q_1 \rangle = \int w_A(p) e^{-2\pi i (u_0 p_0 - u \cdot p)} dp \quad (179)$$

Here, p_1 and q_1 refer to object parameter and sensor states at time t_1 , while q_2 is the sensor state at time t_2 and $u_0 = c(t_2 - t_1)$ represents time change. The change of sensors is given by $u = q_2 - q_1$. The Euclidean norm is effected by

$$p_0(p) = \sqrt{p_1^2 + p_2^2 + \dots + p_n^2} = \sqrt{p \cdot p} \quad (180)$$

Hence, given are measurement data on a collection of individual objects with parameters p_i , each with a characteristic Euclidean norm in Equation (180). The sorting process consists of sorting out the data such that a best fit results in an average target p_A^T , with norm $p_{OA}^T = \sqrt{p_A^T \cdot p_A^T}$. The residue $p_A^N = p_A - p_A^T$ contains what is left of the average measured data p_A after the average target p_A^T has been removed. Notice that the norm property of Equation (180) holds for each individual measured target, but in general $p_{OA} > |p_A|$. This follows from the triangle inequality for norms.

Next, one looks for conditions for which the sorting integral in Equation (179) is maximum. Consider the general phase function

$$f(p') = u_0 p_0(p') - u \cdot p' \quad (181)$$

where the prime indicates the integration variable. The matching condition was previously found to be

$$\nabla f_A = 0 \quad (182)$$

This could be applied directly to Equation (181), giving

$$u_0 \hat{p}_A = u \quad (183)$$

The significance of Equation (183) is not obvious, because p_A is not an object-parameter vector but is the average of input data. Instead, one could expand $f(p')$ in a power series about the desired object p_A^T as follows

$$f(p') = f_A^T + (p'' \cdot \nabla) f_A^T + \frac{1}{2} (p'' \cdot \nabla)^2 f_A^T + \dots \quad (184)$$

where $f_A^T = f(p_A^T)$ and similarly for the higher order terms, i.e., substitute $p = p_A^T$ after differentiations have been performed and finally $p'' = p' - p_A^T$. The matching condition of Equation (182) applied to Equation (184) gives

$$\nabla f_A = \nabla f_A^T + \nabla (p_A^N \cdot \nabla) f_A^T = 0 \quad (185)$$

$$\text{where } p_A^N = p_A - p_A^T \quad (186)$$

It is required that for the selected point p_A^T the function $f(p')$ becomes stationary, giving

$$\nabla f_A^T = 0 \quad (187)$$

which reduces the matching condition of Equation (185) to

$$\nabla (p_A^N \cdot \nabla) f_A^T = 0 \quad (188)$$

All this was shown before with the general discussion of sorting integrals. These conditions are now applied to the Euclidean norm of Equation (180). Equation (187) thus gives $u_0 \hat{p}_A^T = u$. With the definition

$$u_0 = \lambda p_{OA}^T \quad (189)$$

then
$$u = \lambda p_A^T. \quad (190)$$

Comparison of Equations (190) and (183) shows that p_A^T and p_A must be proportional vectors. Equations (189) and (190) are highly significant; they express the equations of motion for the dynamic sorting process. Recall that $u_0 = c \Delta t$ and $u = \Delta q$. Hence, (190) gives the "momentum" $p_A^T = m \frac{\Delta q}{\Delta t} \beta$ for the motion of an elementary particle with energy $p_{OA} = mc \beta$, where $\beta = (1 - (v/c)^2)^{-1/2}$ as usual and $\lambda = \Delta t / m \beta$ is a measure of time-change.

Thus, the conditions of Equations (189) and (190) express the dynamic behavior of the sorting process. Out of the preliminary data p_i , with average p_A , one can sort out the object structure p_A^T which is proportional to p_A but not necessarily equal to it. This depends on the condition of Equation (188) which is discussed next. This equation is satisfied by $p_A^N = 0$ or $p_A = p_A^T$ which is one solution. But as shall be seen, there are other solutions because for the Euclidean norm

$$\nabla (p_A^T \cdot \nabla) f_A^T = 0 \quad (191)$$

By direct differentiation of $f(p)$, we obtain

$$(p' \cdot \nabla)(p'' \cdot \nabla) f(p) = \frac{u_0}{p_0} (p' \cdot p'') - \frac{u_0}{p_0^3} (p' \cdot p)(p'' \cdot p) \quad (192)$$

for any general fixed vectors p' and p'' . By substituting $p'' = p_A^T$ and $p = p_A^T$, Equation (191) follows. Equation (188) may be satisfied by any vector p_A^N proportional to p_A^T : $p_A^N = c p_A^T$. Then $p_A = (c + 1) p_A^T$, after Equation (186). Of particular interest are the cases $c = 0$ and $c = (1 - d)/(1 + d)$, where $p_{OA} = d p_A$, $d > 1$.

The case $c = 0$ corresponds to a decomposition of a Stokes vector into a completely polarized part which has object structure and a completely unpolarized, unstructured, or vacuum part.

The case $c = (1 - d)/(1 + d)$ corresponds to a decomposition of the distributed object into two objects which are orthogonal to each other, i.e., two contrasting or dissimilar objects.

When initially $p_A = 0$, one starts with the so-called vacuous state, the last process results in the creation of two anti-objects: one has the state $(p_{OA}^T ; p_A^T)$, the other $(p_{OA}^T ; -p_A^T)$.

All this is familiar from similar processes in quantum mechanics. In summary, the Euclidean process consists of several possibilities of target decomposition. The first extracts from the available data the single average object which it most likely represents, the residue being "empty" in content or structure. This amounts to:

data = meaningful object + residue, i.e.,
impression of a tree = "tree" + non-differentiated debris.

The second method extracts from the data two meaningful but opposite targets:

aquarium = fish + non-fish (water tank, glass etc.)

These are the first concrete results which are derived from the general theory. The following subsection will give an example of the preceding case.

Example of Stokes Vector Decomposition

The preceding case can be illustrated with a simple case of a Stokes vector. Equation (181) can easily be put in a Stokes vector form:

$$f(p) = \begin{bmatrix} u_o \\ u \end{bmatrix} \cdot \begin{bmatrix} p_o \\ p \end{bmatrix} \quad (193)$$

$$\text{where } p_o = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{p \cdot p} \quad (194)$$

and similarly, since $u_o = \lambda p_{OA}^T$ and $u = \lambda p_A^T$

$$\text{we have: } u_o = \sqrt{u \cdot u} \quad (195)$$

Hence, the scalar product (193) depicts the product of two Stokes vectors:

$$f(p) = u_o p_o - u \cdot p \quad (196)$$

The operation of target decomposition thus reduces, for this case, to a decomposition of Stokes vectors, which takes the familiar form

$$\begin{bmatrix} p_{OA} \\ p_A \end{bmatrix} = \begin{bmatrix} p_{OA}^T \\ p_A^T \end{bmatrix} + \begin{bmatrix} p_O^N \\ o \end{bmatrix} \quad (197)$$

where $p_A = p_A^T$ and $p_{OA} = p_{OA}^T + p_O^N$ and the 'object'-rule for a completely polarized wave $p_{OA}^T = |p_A^T|$ holds.

The above decomposition thus replicates the division of a partially polarized wave into a completely polarized part and a completely unpolarized part.

Another version of Stokes vector decomposition consists in dividing the partially polarized wave into two completely polarized but orthogonal components. This corresponds to a

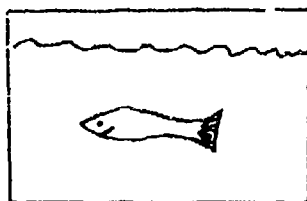
decomposition into a object (c.p. wave I) and anti-object (c.p. wave II) as follows.

$$\begin{bmatrix} p_{oA} \\ p_{AVE} \end{bmatrix} = \begin{bmatrix} p_{oA}^T \\ p_A^T \end{bmatrix}_I + \begin{bmatrix} p_{oA}^T \\ -p_A^T \end{bmatrix}_{II} \quad (198)$$

The first decomposition focuses on separating from the data a "target" or object plus residue (u. p. part). The second version focuses on a distinction between target and non-target.

Example:

Consider an aquarium containing a fish.



In the first case, one concentrates on the object-fish exclusively, such that the rest becomes residue or noise. In the second case, one differentiates the perceived data into "fish" and "nonfish" objects. The "fish" part in this case looms less pronounced in the observer's mind as compared to the first case because the attention is spread over the "fish" and "non-fish" aspects of the scene.

Memory, Storage, and Retrieval

A second area of application relates to data memory, storage and retrieval systems. It is clear that each event must have some special circumstantial characteristic by which it can be recognized. A likely candidate for such a function is the sorting-integral residue $\langle f(p) \rangle_A$.

Now, investigate this possibility. Since

$$f(p) = u_o p_o(p) - u \cdot p \quad (199)$$

$$\text{and } p_o = \sqrt{p \cdot p}, \quad u_o = \lambda p_{oA}^T, \quad u = \lambda p_A^T \quad (200)$$

$$\begin{aligned} \text{we have } \langle f(p) \rangle_A &= \lambda (p_{oA} p_{oA}^T - p_A \cdot p_A^T) = \\ &= \frac{\lambda}{2} [(p_A - p_A^T)^2 - (p_{oA} - p_{oA}^T)^2] + \\ &+ \frac{\lambda}{2} (-p_A^2 - p_A^T^2 + p_{oA}^T^2 + p_{oA}^2) = \\ &= \frac{\lambda}{2} (p_A^N^2 - p_{oA}^N^2 + p_{oA}^2 - p_A^2) = \\ &= \frac{\lambda}{2} (E^2 - E_N^2) = \frac{\lambda}{2} F^2 \end{aligned} \quad (201)$$

where the single target is represented by $p_{oA}^T = |p_A^T|$ and $E^2 = p_{oA}^2 - p_A^2$ is called the excess for the distributed case (if $E = 0$, the target is single). Hence F , defined in Equation (201), is the difference between the excesses of primary data and that of the target residue. If $E_N = 0$, then $E = F$. Hence, F is "stronger" if the primary data are split into contrasting objects rather than into one object plus "environmental debris".

A conceptual graph for the above case is given in Figure 5.

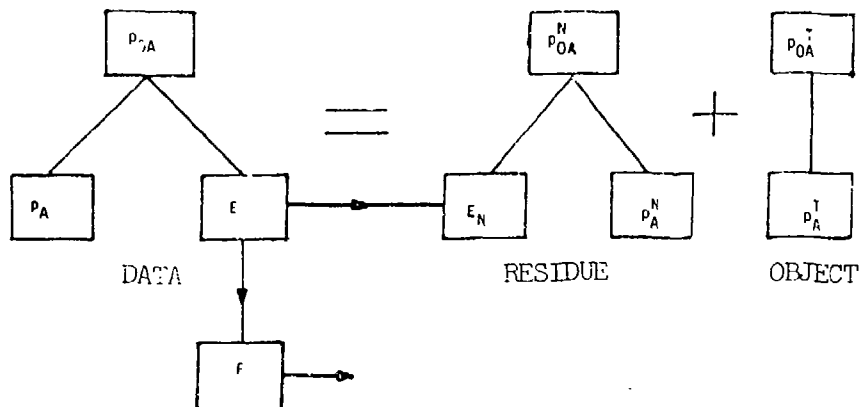


Figure 5. Conceptual Graph of Decomposition of Data into Object Plus Residue

When $F = 0$, the initial data represents a single object and, hence, there is no need for sorting. Hence, F may be considered as a measure of "strength" for the sorting operation. The more successful, the greater F will be, and the greater the chance for storage and recollection for the event.

There is yet another interpretation for F which is derived from the Taylor series expansion of $f(p)$. Recall that

$$f(p') = f_A^T + \frac{1}{2} (p' \cdot \nabla)^2 f_A^T + \dots \quad (202)$$

because $\nabla f_A^T = 0$. Now, it is easy to show that

$$f_A^T = f(p_A^T) = \lambda (p_{OA}^T)^2 - p_A^T{}^2 = 0 \quad (203)$$

From Equation (192) of the previous subsection, with $p'' = p'$:

$$\begin{aligned} (p' \cdot \nabla)^2 f(p) \Big|_A^T &= \lambda p'^2 - \lambda (p' \cdot \hat{p}_A^T)^2 \\ &= \lambda p'^2 (1 - \cos^2 \alpha) = \lambda p'^2 \sin^2 \alpha \end{aligned} \quad (204)$$

$$\text{And hence, } \langle f(p) \rangle_A = \frac{\lambda}{2} F^2 = \frac{\lambda}{2} \langle p'^2 \sin^2 \alpha \rangle \quad (205)$$

$$\text{or } F^2 = \langle p'^2 \sin^2 \alpha \rangle \quad (206)$$

The significance of the angle α is shown in Figure 6.

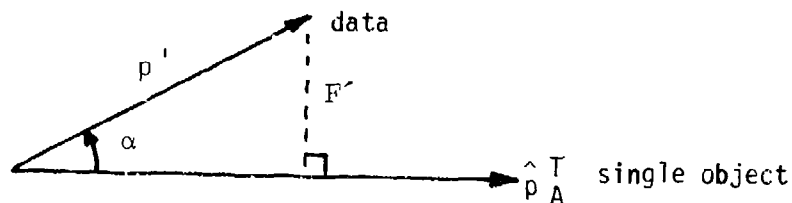


Figure 6. Definition of the Angle α

Since $\langle p' \rangle = p_A$ has the same direction as \hat{p}_A^T , F measures a kind of standard deviation or spread of the initial data from the single average target obtained, in polar coordinates. This information is readily calculated and could be stored for memory and for later retrieval.

Overview of Target Identification Applications

The preceding theory on cognitive systems and processes will have numerous applications. Not every application is obvious from the start in the sense that something new and profitable is immediately apparent. Some systems already in place have a well-established growth and maturity after many years of trial-and-error development through heuristic search. What the theory does in such cases, and where it might be most useful, is to provide an integrating picture. Conversely, by looking at present methodology in various systems, the theory itself will be enriched and can, through feedback into the operating system, provide new insights and new development.

Hence, one may expect a fruitful and productive interaction between the new theory of cognitive systems presented here and certain types of existing expert systems in artificial intelligence (AI) work for example, each case will have its own growth pattern, from very slow to a very rapid and fertile development. The theory will have an integrating effect, such that AI work, linguistics, and even physics, can be shown to have a common ground. This integrating effort can have an enormous impact on the future growth in each of these fields.

At first, the search for good examples and applications will be difficult, mainly because the cognitive investigator lacks the specialized inside knowledge of the field of application which is necessary and essential to make a useful contribution. On the other side, the experts in AI systems cannot be expected to invest the time and effort it takes to follow through in detail on the preceding development in the theory of cognitive systems.

SECTION VIII

SUMMARY

The theory of cognition, as developed recently, has as a primary goal and end-product the realization and identification of a target-objective. In chess, the target-objective is to capture the opponent's king. The sequence of steps which leads to that objective is called the process towards target realization. Given an amount of primary data p_1 , the task consists of organizing this known information into an objective unit which is the goal target itself or a sub-target, as the case may be.

The target objective may be reached through a sequence of subtasks, with defined sub-target or sub-goals. In chess, each subtask is called a move. In cognition, the process which leads to the realization of the target-objective is called sorting. Each task or sub-task is defined by the primary data, p_1 , the distribution function $w(p_1, p_2, \dots)$ associated with the data and a set of constant dynamic factors. The sorting process consists of sorting out the primary data in some optimal fashion to find a best approach that leads to realizing the target-objective.

All this is expressed mathematically by the sorting integral. The processes which lead to optimizing the sorting integral are the same processes which lead to realization of the target-objective. Hence, there is a one-to-one correspondence between the processes which optimize the sorting integral and the physical processes which lead to target identification or realization, as the case may be. The conditions which have to be satisfied such that the sorting integral is maximum are called target matching conditions. These determine the direction in which the process should move in order to reach the target-objective. Hence, they determine the dynamics of the sorting operation.

All this must sound familiar to the physicist, because the dynamics of a closed physical system also is determined by a process of optimization of an integral. There, the action-integral represents the physical process and it determines the so-called equations of motion for the system. The action-integral originally was developed in the 19th century by Hamilton, Jacobi, Lagrange, and others in an attempt to unify classical mechanics. There is a distinct difference between action integrals in the classical theory and the so-called Feynman path-integral approach. Whereas the former maximizes or minimizes a physical quantity such as the time-interval a ray takes to traverse the distance between two fixed points in space, the latter is concerned with an integral over a phase-function which represents the classical action. The principal value of the integration process will be from those points where the phase is stationary. Hence, the classical path is, to a large extent, defined by the stationary points of the phase function.

This theme carries over to cognition. But unlike the path-integral, the sorting integral has a distribution function in the integrand which represents and limits target-information. One can explain this difference by observing that the cognitive process is guided by limited and specified knowledge which the observer has available from the target, whereas the physical state process has no such preference. The presence of the distribution function $w(p)$ in the integrand makes the sorting integral behave more like a correlation function.

A best fit to desired objectives is reached when the correlation function is maximized. One has to keep these distinctions in mind if one compares path-integrals with sorting integrals. Also, the mathematical evaluation becomes quite different for the two cases. The Feynman path-integral cannot be considered as a limiting case of the sorting integral when the distribution function becomes uniform.

Despite these differences, there are remarkable similarities which make a comparison useful to guide one's understanding of

cognitive operations. In both cases, the essence of the physical state of the problem at hand is captured by a single compact mathematical formulation. In physics, this function is called the Lagrangian, or equivalently the Hamiltonian, function. In cognition, the essence of the sorting procedure is captured by a norm function or noun. One can show that the object Hamiltonian, under suitable conditions, satisfies the norm-function requirements.*

The essence of the physical state is that it provides us with a "unit of apperception", to use a Kantian phrase. The rest of the world is as if it were blocked off and what is under consideration is a small part, a micro-world, which nevertheless for itself has the property of wholeness and atomicity. In quantum theory, such a physical system is called a "pure state" capable of being represented by a single wave function. This is in contrast to an ensemble of pure states, which is called a target mixture.

In cognition, one speaks of a concept of an object or target which is distinct from an assembly of objects. It turns out that the above-mentioned norm property provides the key ingredient by which the concept attains its character of uniqueness, atomicity, and irreducibility.

If knowledge progresses through stages of concept formulation, then one has here at one's disposal a mathematical tool for such a development. Target identification then consists of finding the proper concept for the object which optimally fits a set of measured data.

Thus, a bridge is laid between what is known in perception, AI theory, computer work, quantum theory, radar sensing mechanisms, and a general theory of cognitive states and processes.

* Reference 16, pp. 211-212.

Applications can be found with memory and retrieval systems, problem solving techniques, target identification and pattern recognition, robotics, and linguistics research just to mention a few areas.

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